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On the Numerical Solution of Sturm-Liouville Differential Equations

1. Introduction. It is the purpose of this paper to use the well-known relation that exists between a Sturm-Liouville differential equation together with its boundary conditions and normalization condition and a problem in the calculus of variations, to obtain an algebraic characteristic value problem from whose solution we may obtain approximations to the characteristic values and characteristic functions of the Sturm-Liouville equation. That is, we shall obtain approximations to the values of λ and the functions $y(x)$ satisfying the equation

$$y'' + (q(x) + \lambda r(x))y = 0, \tag{1.1}$$

such that the boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \beta_1 y(b) + \beta_2 y'(b) &= 0 \end{aligned} \tag{1.2}$$

and the normalization condition

$$\int_a^b ry^2 dx = 1 \tag{1.3}$$

are satisfied, where the prime denotes the derivative of $y(x)$ with respect to x, q and r being functions of x, and the latter function is required to be positive over the interval

$$a \leq x \leq b.$$

Equations of the form

$$(py')' + qy + \lambda ry = 0$$

may be reduced to that of the form of equation (1.1) by a transformation of the independent variable.

The equations (1.1) and (1.2) themselves may be approximated by a set of linear algebraic equations as follows (cf., Hildebrand [1]). Let us use a set of discrete values of x

$$x_i = a + ih \quad (i = 0, 1, \cdots, N + 1)$$

$$h = \frac{b - a}{N + 1}$$

and define

$$y_i = y(x_i).$$

Then we may approximate $y'(x_i) = y_i'$ and $y''(x_i) = y_i''$ by means of finite differences of the y_i. We would then obtain a set of equations of the form

$$L_{ij}y_j = \lambda r_{ij}y_i \tag{1.4}$$

for the unknowns λ and y_i. That is, we would thus reduce the system of equations (1.1) and (1.2) to an algebraic characteristic value problem. However the matrix L_{ij} would not in general be symmetric. Indeed as the order of the differences used to replace the derivatives increases, the asymmetry of the matrix L_{ij} increases. This asymmetry occurs because the same type of difference formula cannot be used at the ends of the interval (a,b) as in the interior of the interval.

If the matrix L_{ij} is not symmetric, the reality of the values of λ cannot be guaranteed. Moreover the determination of the values of λ such that equation (1.4) has a solution is a much greater computational problem when L_{ij} is an asymmetric matrix than when it is a symmetric one. The methods that will be used in the sequel to replace the system of equations (1.1) to (1.3) by an algebraic characteristic value problem will always lead to symmetric matrices.

2. The Variational Problem. It is well known that equation (1.1) is the Euler condition that the integral

$$\int_a^b ((y')^2 - qy^2)dx \tag{2.1}$$

be an extremal where the functions $y(x)$ satisfy equation (1.3).

Hence the system of equations (1.1) to (1.3) is equivalent to finding functions $y(x)$ subject to equations (1.2) such that the integral

$$I(y) = \int_a^b [(y')^2 - qy^2]dx - \lambda \int_a^b ry^2 dx \tag{2.2}$$

is an extremum where λ is a Lagrange multiplier and the functions $y(x)$ are restricted to satisfy the conditions (1.2).

Instead of dealing with the differential equation (1.1) we deal with the integral (2.2). There are two methods that we shall consider: (1) approximating this integral by using quadrature and finite difference formulas (Method I), and (2) restricting the class of admissible functions to be made up of polynomial arcs (Method II). The latter method is of course a variant of the Rayleigh-Ritz method. However it differs from that method in that it uses a set of polynomials, each of degree m, say, and each one defined over an interval $x_i \leq x < x_{i+1}$, to represent an admissible function $y(x)$ instead of an expansion of $y(x)$ in terms of a convenient orthonormal set of functions. The latter representation is required to hold over the entire interval (a,b) and many terms may be needed because of the behavior of $y(x)$ on a few of the intervals $x_i \leq x < x_{i+1}$; whereas in the former representation we may use intervals of varying sizes or allow the admissible function to be made up of polynomial arcs, where the degree of each arc need not be the same.

3. Method I. In the discussion of this method we shall restrict ourselves to the case where $\alpha_2 = \beta_2 = 0$ in equations (1.2). The discussion may be readily modified to take care of more general boundary conditions.

The first step is to obtain a finite difference approximation to (2.2). To do this we introduce the quadrature formula

$$\int_a^b \phi(x)dx = \sum_{i=0}^{N+1} w_i\phi_i + \epsilon_1, \tag{3.1}$$

where ϵ_1 is the error in this formula. The weighting factors w_i for the functional values ϕ_i are to be determined later. We also introduce the formula

$$y_i' = h^{-1} \sum_{j=0}^{N+1} a_{ij}y_j + \eta_i \quad (i = 0, 1, \cdots, N+1) \tag{3.2}$$

which is a finite difference approximation to the first derivative at x_i in terms of the functional values y_i. The values of a_{ij} are to be determined later also.

Using (3.1) and (3.2) we approximate the first integral in (2.2) obtaining

$$\begin{aligned}\int_a^b [(y')^2 - qy^2]dx &= \sum_{i=0}^{N+1} w_i[(y_i')^2 - q_iy_i^2] + \epsilon_1 \\ &= \sum_{i=0}^{N+1} w_i[h^{-1} \sum_{j=0}^{N+1} a_{ij}y_j + \eta_i]^2 - \sum_{i=0}^{N+1} w_iq_iy_i^2 + \epsilon_1 \\ &= \sum_{j,k=0}^{N+1} [h^{-2} \sum_{i=0}^{N+1} w_ia_{ij}a_{ik} - w_jq_j\delta_{jk}]y_jy_k + (\epsilon_1 + \epsilon_2)\end{aligned}$$

where

$$\epsilon_2 = \sum_{i=0}^{N+1} w_i[2\eta_ih^{-1} \sum_{j=0}^{N+1} a_{ij}y_j + \eta_i^2]$$

and δ_{jk} is the Kronecker delta. Our equation becomes

$$\int_a^b [(y')^2 - qy^2]dx = \sum_{j,k=0}^{N+1} L_{jk}y_jy_k + (\epsilon_1 + \epsilon_2) \tag{3.3}$$

if we set

$$L_{jk} = h^{-2} \sum_{i=0}^{N+1} w_ia_{ij}a_{ik} - w_jq_j\delta_{jk}. \tag{3.4}$$

Likewise the second integral in (2.2) becomes

$$\int_a^b ry^2dx = \sum_{j,k=0}^{N+1} w_jr_j\delta_{jk}y_jy_k + \epsilon_3. \tag{3.5}$$

Thus $I(y)$ is approximated by the quadratic form

$$F = \sum_{j,k=0}^{N+1} L_{jk}y_jy_k - \lambda^* \sum_{j,k=0}^{N+1} w_jr_j\delta_{jk}y_jy_k, \tag{3.6}$$

and if we insert the boundary conditions $y_0 = y_{N+1} = 0$ we obtain simply

$$F = \sum_{j,k=1}^{N} L_{jk} y_j y_k - \lambda^* \sum_{j,k=1}^{N} w_j r_j \delta_{jk} y_j y_k.$$

The second step in our numerical method is to minimize the quadratic form. This is a simple problem in the calculus and the necessary condition for a minimum is that

$$\frac{\partial F}{\partial y_j} = \sum_{k=1}^{N} L_{jk} y_k - \lambda^* \sum_{k=1}^{N} w_j r_j \delta_{jk} y_k = 0,$$

for $j = 1, \cdots, N$. This system of linear equations can be written in matrix notation

(3.7) $$Ly - \lambda^* WRy = 0,$$

where L is a symmetric $N \times N$ matrix and WR is a diagonal one.

In order to put (3.7) in a form analogous to (1.1) we multiply through by $-1/w_j$, and if we define t_{jk} to be

$$t_{jk} = -\,(1/w_j)L_{jk},$$

we can write

$$\sum_{k=1}^{N} t_{jk} y_k + \lambda^* r_j y_j = 0,$$

which becomes

$$Ty + \lambda^* Ry = 0,$$

in matrix form. Using (3.4) and setting

$$G_j = 1/w_j h^2,$$

we can write t_{jk} as

(3.8) $$t_{jk} = -\,G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}.$$

We note that L and T are functions of w_j and a_{ij} from the approximation formulas (3.1) and (3.2) respectively. In the next section a method will be discussed for selecting w_j and a_{ij} so as to have T approximate the differential operator

$$D_2 \equiv \frac{d^2}{dx^2} + q,$$

to a specified order of approximation.

4. The Order of Approximation of the Method. It was stated in the last section that the matrix operator T is to be an approximation to the differential operator D_2. We would like to specify the order of this approximation. However, before doing so let us define what we mean by a certain "order of approximation" in this case.

In order to compare a differential operator with a matrix operator we need to arrive at a common basis for comparison. To do this consider the vector

$$v_1 = \begin{bmatrix} q_1y_1 + y_1'' \\ \cdot \\ \cdot \\ \cdot \\ q_jy_j + y_j'' \\ \cdot \\ \cdot \\ \cdot \\ q_Ny_N + y_N'' \end{bmatrix},$$

whose components are obtained by evaluating D_2y at the discrete set of mesh points x_j, $j = 1, \cdots, N$. Also consider the vector

$$v_2 = \begin{bmatrix} \sum_{k=1}^{N} t_{1k}y_k \\ \cdot \\ \cdot \\ \cdot \\ \sum_{k=1}^{N} t_{jk}y_k \\ \cdot \\ \cdot \\ \cdot \\ \sum_{k=1}^{N} t_{Nk}y_k \end{bmatrix}$$

Definition. If

$$\sum_{k=1}^{N} t_{jk}y_k = q_jy_j + y_j'' + 0(h^\sigma) \tag{4.1}$$

for $j = 1, \cdots, N$, then the matrix T approximates the differential operator D_2 to order σ.

This definition can be used as a basis for the selection of w_j and a_{ij} for formulas (3.1) and (3.2) respectively. To see this let us require that w_j and a_{ij} be chosen in such a way that the numerical method is of order σ. Let us further require that

$$y_i' = h^{-1} \sum_{k=0}^{N+1} a_{ik}y_k + 0(h^\sigma), \tag{4.2}$$

i.e., that (3.2) be an approximation formula also of order σ. This means that the a_{ik} must satisfy the equations

$$\sum_{k=0}^{N+1} a_{ik}(k - i)^\mu = \delta_{1\mu} \quad (\mu = 0, 1, \cdots, \sigma). \tag{4.3}$$

For, if we expand y_k about y_i, we get

$$y_k = \sum_{\mu=0}^{\infty} (k - i)^\mu y_i^{(\mu)} \frac{h^\mu}{\mu!}$$

and thus, in (4.2),

$$h^{-1} \sum_{k=0}^{N+1} a_{ik} y_k = \sum_{\mu=0}^{\infty} \left[\sum_{k=0}^{N+1} a_{ik}(k - i)^\mu \right] y_i^{(\mu)} \frac{h^{\mu-1}}{\mu!}$$

from which (4.3) follows.

Using the same procedure, and recalling that $y_0 = y_{N+1} = 0$, we can write the left side of equation (4.1) as

$$\sum_{k=1}^{N} t_{jk} y_k = \sum_{0}^{N+1} t_{jk} y_k = \left\{ y_j \left[\sum_{k=0}^{N+1} t_{jk} \right] + y_j' h \left[\sum_{k=0}^{N+1} (k - j) t_{jk} \right] \right.$$
$$\left. + y_j'' \frac{h^2}{2!} \left[\sum_{k=0}^{N+1} (k - j)^2 t_{jk} \right] + \cdots \right\}.$$

Thus we have from equation (3.8)

$$\sum_{k=1}^{N} t_{jk} y_k = y_j \left[\sum_{k=0}^{N+1} (- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) \right]$$
$$+ y_j' h \left[\sum_{k=0}^{N+1} (k - j)(- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) \right]$$
$$+ y_j'' \frac{h^2}{2!} \left[\sum_{k=0}^{N+1} (k - j)^2 (- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) \right]$$
$$+ \cdots.$$

Now we are in a position to treat (4.1) as an identity in y_i and its derivatives by using the last equation. By equating coefficients of y_i and its derivatives up to $y^{(\sigma+1)}$, we obtain

$$\sum_{k=0}^{N+1} (- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) = q_j,$$

$$\sum_{k=0}^{N+1} (k - j)(- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) = 0,$$

$$\frac{h^2}{2} \sum_{k=0}^{N+1} (k - j)^2 (- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) = 1,$$

and

$$\sum_{k=0}^{N+1} (k - j)^r (- G_j \sum_{i=0}^{N+1} w_i a_{ij} a_{ik} + q_j \delta_{jk}) = 0,$$

where $\tau = 3, \cdots, \sigma + 1$. Since

$$\sum_{k=0}^{N+1} q_j\delta_{jk} = q_j,$$

and

$$(k - j)\delta_{jk} = 0,$$

and using the definition of G_j, these become

$$\sum_{i=0}^{N+1} w_i a_{ij} \sum_{k=0}^{N+1} a_{ik} = 0,$$

$$\sum_{i=0}^{N+1} w_i a_{ij} \sum_{k=0}^{N+1} a_{ik}(k - j) = 0,$$

$$\sum_{i=0}^{N+1} w_i a_{ij} \sum_{k=0}^{N+1} a_{ik}(k - j)^2 = -2w_j,$$

and

$$\sum_{i=0}^{N+1} w_i a_{ij} \sum_{k=0}^{N+1} a_{ik}(k - j)^\tau = 0,$$

where $\tau = 3, \cdots, \sigma + 1$.

A further simplification results if we use equation (4.3), i.e., if we make use of the fact that the a_{ij} give us an approximation to the derivative of order σ. For we can write

$$\sum_{k=0}^{N+1} a_{ik}(k - j)^\mu = \sum_{k=0}^{N+1} a_{ik}[(k - i) + (i - j)]^\mu$$

$$= \sum_{k=0}^{N+1} a_{ik}[(k - i)^\mu + \mu(k - i)^{\mu-1}(i - j) + \cdots + \mu(k - i)(i - j)^{\mu-1} + (i - j)^\mu]$$

$$= \mu(i - j)^{\mu-1}.$$

With this result our equations become

$$\sum_{i=0}^{N+1} w_i a_{ij} = 0,$$

$$\sum_{i=0}^{N+1} w_i a_{ij}(j - i) = w_j,$$

and

$$\sum_{i=0}^{N+1} w_i a_{ij}(j - i)^\tau = 0 \quad (\tau = 2, \cdots, \sigma),$$

or simply

$$\sum_{i=0}^{N+1} w_i a_{ij}(j-i)^\tau = w_j \delta_{1\tau} \quad (\tau = 0, \cdots, \sigma). \tag{4.4}$$

In these equations $j = 1, \cdots, N$.

Thus if we select a_{ij} and w_j satisfying equations (4.3) and (4.4) our method will give an approximation of order σ.

5. The First Order Approximation. In this case $\sigma = 1$ and equations (4.3) and (4.4) become

$$\begin{gathered} \sum_{k=0}^{N+1} a_{ik} = 0, \\ \sum_{k=0}^{N+1} a_{ik}(k-i) = 1 \quad (i = 0 \cdots N+1), \end{gathered} \tag{5.1}$$

and

$$\begin{gathered} \sum_{i=0}^{N+1} w_i a_{ij} = 0, \\ \sum_{i=0}^{N+1} w_i a_{ij}(j-i) = w_j \quad (j = 1, \cdots, N), \end{gathered} \tag{5.2}$$

respectively. Note that equations (5.1) give conditions on the rows of the matrix (a_{ik}) whereas equations (5.2) give conditions on the columns (except for the first and last) of the same matrix.

It is easily verified that the matrix

$$(a_{ik}) = \left[\begin{array}{cccccccccc} -1 & 1 & 0 & 0 & 0 & \cdot & & & & \\ 0 & -1 & 1 & 0 & 0 & \cdot & & & & \\ 0 & 0 & -1 & 1 & 0 & \cdot & & & & \\ 0 & 0 & 0 & -1 & 1 & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & -1 & 1 & 0 & 0 \\ & & & & & \cdot & 0 & -1 & 1 & 0 \\ & & & & & \cdot & 0 & 0 & -1 & 1 \\ & & & & & \cdot & 0 & 0 & -1 & 1 \end{array}\right], \tag{5.3}$$

and w_i satisfying

$$\begin{gathered} w_{N+1} = 0, \\ w_i = w_{i+1} \quad (i = 0, \cdots, N-1) \end{gathered} \tag{5.4}$$

constitute a solution of (5.1) and (5.2), i.e., equations (5.3) and (5.4) give us a_{ik} and w_i for the first order case.

Equation (5.3) states that we use the simple first order forward difference formula

$$y_i' = (y_{i+1} - y_i)/h + [y_i''(h/2!) + y_i^{(3)}(h^2/3!) + \cdots],$$

for $i = 0, \cdots, N$, and the simple first order backward difference formula

$$y_i' = (y_i - y_{i-1})/h - [y_i''(h/2!) - y_i^{(3)}(h^2/3!) + \cdots],$$

for $i = N + 1$. Equations (5.4) indicate that the quadrature formula to use is the simple first order "Rectangle Rule"

$$\int_a^b f(x)dx = w_1 \sum_{i=0}^{N} f_i - [f_1'(h/2) + \cdots]. \tag{5.5}$$

Obviously

$$w_1 = (b - a)/(N + 1) = h,$$

since (5.5) must hold for $f(x) \equiv 1$.

6. The Second Order Approximation. In this case $\sigma = 2$ and equations (4.3) and (4.4) become

$$\begin{gathered} \sum_{k=0}^{N+1} a_{ik} = 0, \\ \sum_{k=0}^{N+1} a_{ik}(k - i) = 1, \\ \sum_{k=0}^{N+1} a_{ik}(k - i)^2 = 0 \quad (i = 0, \cdots, N + 1), \end{gathered} \tag{6.1}$$

and

$$\begin{gathered} \sum_{i=0}^{N+1} w_i a_{ij} = 0, \\ \sum_{i=0}^{N+1} w_i a_{ij}(j - i) = w_j, \\ \sum_{i=0}^{N+1} w_i a_{ij}(j - i)^2 = 0 \quad (j = 1, \cdots, N), \end{gathered} \tag{6.2}$$

respectively.

It has been shown by Gregory [2] that when N is odd a solution to equations (6.1) and (6.2) is given by the matrix (a_{ik}) of the form

$$(a_{ik}) = \left[\begin{array}{cccccccccc} -3/2 & 2 & -1/2 & 0 & 0 & 0 & 0 & & & \\ -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & & & \\ 1/4 & -1 & 0 & 1 & -1/4 & 0 & 0 & & & \\ 0 & 0 & -1/2 & 0 & 1/2 & 0 & 0 & & & \\ 0 & 0 & 1/4 & -1 & 0 & 1 & -1/4 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & 0 & -1/2 & 0 & 1/2 & 0 & 0 \\ & & & & 0 & 1/4 & -1 & 0 & 1 & -1/4 \\ & & & & 0 & 0 & 0 & -1/2 & 0 & 1/2 \\ & & & & 0 & 0 & 0 & 1/2 & -2 & 3/2 \end{array}\right], \tag{6.3}$$

and the w_i are given by

$$
\begin{aligned}
w_1 &= 4w_0, \\
w_2 &= 2w_0, \\
w_3 &= 4w_0, \\
&\;\;\vdots \\
w_{N-1} &= 2w_0, \\
w_N &= 4w_0, \\
w_{N+1} &= w_0.
\end{aligned}
\tag{6.4}
$$

If N is even the solutions differ only in the last four rows of the matrix (a_{ik}) and the last four values of w_i, i.e.,

$$
(a_{ik}) = \begin{bmatrix}
-3/2 & 2 & -1/2 & 0 & 0 & 0 & & & \\
-1/2 & 0 & 1/2 & 0 & 0 & 0 & & & \\
1/4 & -1 & 0 & 1 & -1/4 & 0 & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & 0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 \\
 & & 0 & 4/17 & -16/17 & -3/34 & 18/17 & -9/34 & 0 \\
 & & 0 & 0 & 0 & -1/3 & -1/2 & 1 & -1/6 \\
 & & 0 & 0 & 0 & -1/6 & 0 & -1/2 & 2/3 \\
 & & 0 & 0 & 0 & 1/3 & -1/2 & -1 & 7/6
\end{bmatrix},
\tag{6.5}
$$

and

$$
\begin{aligned}
w_1 &= 4w_0, \\
w_2 &= 2w_0, \\
&\;\;\vdots \\
w_{N-4} &= 2w_0, \\
w_{N-3} &= 4w_0, \\
w_{N-2} &= \frac{17}{8}\,w_0, \\
w_{N-1} &= \frac{27}{8}\,w_0, \\
w_N &= \frac{27}{8}\,w_0, \\
w_{N+1} &= \frac{9}{8}\,w_0.
\end{aligned}
\tag{6.6}
$$

Equations (6.6) can be interpreted as a quadrature formula consisting of Simpson's Rule for all but the last three intervals and the "3/8 Rule" for the last three intervals. This special treatment at the end is due to the fact that Simpson's Rule can only be used with an odd number of mesh points and here N is even.

In a later section the solutions of equations (3.7), where L_{ij} is defined by equation (3.4) for the first and second order approximations, will be compared

to the characteristic values and values of the characteristic functions at the mesh points of the Sturm-Liouville system described by the equation

$$\frac{d^2y}{dx^2} + \lambda y = 0 \tag{6.7}$$

with the boundary conditions

$$y(0) = y(1) = 0 \tag{6.8}$$

and the normalization condition

$$\int y^2\,dx = 1. \tag{6.9}$$

This simple example for which the solutions are known to be

$$\lambda_n = \pi^2 n^2 \tag{6.10}$$

$$y_n = \sqrt{2}\sin \pi n x \tag{6.11}$$

will be used in the discussion of the effectiveness of Methods I and II.

7. Method II. We now turn our attention to the derivation of the algebraic characteristic value problem associated with Method II. In this method, instead of approximating the quadratures occurring in (2.2) by mechanical formulas, the unknown solution $y(x)$ itself is replaced by an approximating function $Y(x)$. The approximating function $Y(x)$ is defined in terms of $N + 2$ parameters y_j in a manner such that $Y(x)$ assumes the value y_j at the mesh point x_j, i.e., $Y(x_j) = y_j$ ($j = 0, \cdots, N + 1$). More specifically, in each interval $x_i \leq x < x_{i+1}$ the approximating function $Y(x)$ is defined to be a polynomial $Y_i(x)$ of degree m passing through y_i and m other prescribed nearby ordinates. The boundary conditions (1.2) determine two of the ordinates in terms of the others. Then the integral (cf., equation (2.2))

$$I(Y) = \int_a^b (Y'^2 - qY^2 - \lambda r Y^2)dx \tag{7.1}$$

is a function of the values y_j assumed by $Y(x)$ at N distinct selected points. Requiring that the approximating function $Y(x)$ be an extremal of (7.1) leads, without further approximations, to an algebraic characteristic value problem determining N sets of values for λ and the ordinates y_j.

The polynomials $Y_i(x)$ defining the function $Y(x)$ are given by the Lagrange interpolation formula

$$Y_i(x) = \frac{(x - x_{i_1}) \cdots (x - x_{i_m})}{(x_{i_0} - x_{i_1}) \cdots (x_{i_0} - x_{i_m})} y_{i_0} + \cdots + \frac{(x - x_{i_0}) \cdots (x - x_{i_{m-1}})}{(x_{i_m} - x_{i_0}) \cdots (x_{i_m} - x_{i_{m-1}})} y_{i_m} \tag{7.2}$$

where $i_k = i + k - l_i$ $(k = 0, 1, \cdots, m)$ and the l_i are chosen such that $Y(x)$ is given by "forward" or "backward" interpolation formulas near the end points and by nearly "central" formulas at interior points and such that the interpolation polynomials are chosen symmetrically with respect to the midpoint $(a + b)/2$. More specifically, in obtaining the numerical results reported in this paper N is odd and the choice of the ordinates through which each $Y_i(x)$ is required to pass is equivalent to the following rules determining the single integer l_i

$$(7.3)\qquad \begin{array}{ll} l_i = i & \text{for } i \le \tfrac{1}{2}m - 1 \\ \tfrac{1}{2}m - 2 < l_i \le \tfrac{1}{2}m - 1 & \text{for } \tfrac{1}{2}m - 1 < i < \tfrac{1}{2}N \\ \tfrac{1}{2}m \le l_i < \tfrac{1}{2}m + 1 & \text{for } \tfrac{1}{2}N < i < N + 1 - \tfrac{1}{2}m \\ l_i = i + m - N - 1 & \text{for } i \ge N + 1 - \tfrac{1}{2}m. \end{array}$$

Henceforth we shall assume that the mesh points x_i are equally spaced with $x_0 = a$, $x_{N+1} = b$ and $x_{i+1} - x_i = h = (b - a)/(N + 1)$.

Since each $Y_i(x)$ depends linearly upon the ordinates y_j, its derivatives do also. Hence we may write for the rth derivative of $Y_i(x)$ at x_i

$$(7.4)\qquad \frac{h^r}{r!} Y_i^{(r)}(x_i) = \sum_{j=1}^{N} \alpha^i{}_{rj} y_j.$$

For fixed values of i and r, at most $m + 1$ of the coefficients $\alpha^i{}_{rj}$ are different from zero. The summation in (7.4) extends from $j = 1$ to $j = N$ since y_0 and y_{N+1} may be determined in terms of the others by the boundary conditions (1.2). The $\alpha^i{}_{rj}$ thus will in general depend upon the boundary conditions. In the special case where $\alpha_2 = \beta_2 = 0$, however, we have $y_0 = y_{N+1} = 0$, and the $\alpha^i{}_{rj}$ are rational numbers which can be expressed as integers divided by $m!$ From (7.4) it follows by Taylor's theorem that

$$(7.5)\qquad Y_i(x) = \sum_{r=0}^{m} \sum_{j=1}^{N} h^{-r} \alpha^i{}_{rj} y_j (x - x_i)^r.$$

We shall discuss the determination of the quantities $\alpha^i{}_{rj}$ in the next section. In the remainder of this section we assume that they are known and evaluate equation (7.1) for

$$(7.6)\qquad Y(x) = Y_i(x) \quad x_i \le x < x_{i+1}$$

where $Y_i(x)$ is given by equation (7.5).

Substituting equation (7.6) into (7.1) gives

$$(7.7)\qquad \begin{aligned} I(Y) = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \{[\sum_r \sum_j r h^{-r} \alpha^i{}_{rj} y_j (x - x_i)^{r-1}]^2 \\ - [q(x) + \lambda r(x)][\sum_r \sum_j h^{-r} \alpha^i{}_{rj} y_j (x - x_i)^r]^2\} dx. \end{aligned}$$

If we define

$$P^i_{rt} = \frac{rt}{h^{r+t+1}} \int_{x_i}^{x_{i+1}} (x - x_i)^{r+t-2}\, dx, \tag{7.8}$$

$$Q^i_{rt} = \frac{1}{h^{r+t+1}} \int_{x_i}^{x_{i+1}} q(x)(x - x_i)^{r+t}\, dx, \tag{7.9}$$

$$R^i_{rt} = \frac{1}{h^{r+t+1}} \int_{x_i}^{x_{i+1}} r(x)(x - x_i)^{r+t}\, dx, \tag{7.10}$$

$$L_{jk} = \sum_{i=0}^{N} \sum_{r,t=0}^{m} \alpha^i_{rj}(P^i_{rt} - Q^i_{rt})\alpha^i_{tj}, \tag{7.11}$$

$$D_{jk} = \sum_{i=0}^{N} \sum_{r,t=0}^{m} \alpha^i_{rj} R^i_{rt} \alpha^i_{tk}, \tag{7.12}$$

equation (7.7) becomes

$$I(Y) = \sum_{j,k=0}^{N} (L_{jk} - \lambda D_{jk}) y_j y_k.$$

The extrema of $I(Y)$ are determined by the numbers y_j satisfying

$$\sum_{k=1}^{N} L_{jk} y_k = \lambda \sum_{k=1}^{N} D_{jk} y_k. \tag{7.13}$$

for $j = 1, \cdots, N$. This equation may be written in matrix notation as

$$LY = \lambda DY \tag{7.14}$$

when Y is the matrix of a single column:

$$Y = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix}.$$

8. Determination of α^i_{rj}. Formulas for the determination of the numbers α^i_{rj} may be obtained by differentiation of equation (7.2) and using the boundary conditions. However, as will be shown below we may also determine these quantities by a double induction, one on the index i and one on the degree of the polynomials involved. For simplicity in the ensuing discussion we assume that we are dealing with the special boundary conditions

$$y_0 = y_{N+1} = 0. \tag{8.1}$$

Similar methods apply in the case of general boundary conditions.

We first discuss the inductive procedure we use to determine the $\alpha^0{}_{rj}$ which are defined by the equation

$$\frac{h^r Y_0^{(r)}(x_0)}{r!} = \sum_{j=0}^{N+1} \alpha^0{}_{rj} y_j \tag{8.2}$$

where $Y_0^{(r)}(x_0)$ is the rth derivative of the polynomial $Y_0(x)$ evaluated at $x = x_0$. By definition $Y_0(x)$ is the mth degree Lagrange polynomial passing through $y_0, y_1, \cdots, y_m$.

Let $P_s(x)$ be the sth degree Lagrange polynomial passing through $y_0, y_1, \cdots, y_s$. Then

$$Y_0(x) = P_m(x).$$

We note that

$$P_0(x) \equiv y_0. \tag{8.3}$$

In general we have

$$P_s(x) = \sum_{k=0}^{s} p_{sk}(x) y_k \tag{8.4}$$

where

$$p_{sk}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_s)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_s)}. \tag{8.5}$$

We may also write

$$P_s(x) = \sum_{r=0}^{s} \frac{P_s^{(r)}(x_0)}{r!} (x - x_0)^r$$

where $P_s^{(r)}(x_0)$ is the rth derivative of $P_s(x)$ evaluated at $x = x_0$. We may further write

$$\frac{h^r}{r!} P_s^{(r)}(x_0) = \sum_{k=0}^{N+1} \beta^s{}_{rk} y_k. \tag{8.6}$$

Then

$$\alpha^0{}_{rj} = \beta^m{}_{rj} \tag{8.7}$$

and

$$\beta^s_{rj} = \frac{h^r}{r!} p_{sj}^{(r)}(x_0). \tag{8.8}$$

It follows from equations (8.4) and (8.5) that

$$p_{s+1\,k} = \frac{x - x_{s+1}}{x_k - x_{s+1}} p_{sk} \quad k < s + 1 \tag{8.9}$$

$$(8.10)\quad p_{s+1\,s+1} = \frac{(x - x_s)}{(x_{s+1} - x_s)} \frac{(x_s - x_0)(x_s - x_1) \cdots (x_s - x_{s-1})}{(x_{s+1} - x_0)(x_{s+1} - x_1) \cdots (x_{s+1} - x_{s-1})} p_{ss}(x)$$

$$= \frac{(x - x_s)}{h(s + 1)} p_{ss}(x),$$

the last equation holding when the intervals are equal in length.

Differentiating equations (8.9) and (8.10) r times we have

$$p^{(r)}_{s+1\,k}(x) = \frac{x - x_{s+1}}{x_k - x_{s+1}} p^{(r)}_{sk}(x) + \frac{r}{x_k - x_{s+1}} p^{(r-1)}_{sk}(x)$$

$$p^{(r)}_{s+1\,s+1}(x) = \frac{x - x_s}{h(s + 1)} p^{(r)}_{ss}(x) + \frac{r}{h(s + 1)} p^{(r-1)}_{ss}(x).$$

In view of equation (8.8) these equations may be written as

$$(8.11)\qquad \beta^{s+1}_{r\,k} = \frac{s + 1}{s + 1 - k} \beta^s_{rk} - \frac{1}{s + 1 - k} \beta^s_{r-1\,k}$$

$$\beta^{s+1}_{r\,s+1} = -\frac{s}{s + 1} \beta^s_{rs} + \frac{1}{s + 1} \beta^s_{r-1\,s}.$$

It follows from equation (8.3) that

$$\beta^0_{00} = 1$$

$$\beta^0_{rk} = 0 \quad r \neq 0,\ k \neq 0.$$

These equations together with equations (8.11) define β^s_{rj} and hence α^0_{rj} in view of equation (8.7).

To obtain formulas by which the α^i_{rj} may be computed for $i > 0$ we consider first the case in which $Y_i(x)$ and $Y_{i+1}(x)$ are the same polynomial. The α^i_{rj} and the α^{i+1}_{rj} then determine the derivatives of this polynomial at x_i and x_{i+1} respectively. Since $Y_i(x) = Y_{i+1}(x)$ we have in particular

$$\frac{h^r}{r!} Y^{(r)}_i(x_{i+1}) = \frac{h^r}{r!} Y^{(r)}_{i+1}(x_{i+1}).$$

Now

$$Y^{(r)}_i(x_{i+1}) = \sum_{t=r}^{m} Y^{(t)}_i(x_i) \frac{h^{t-r}}{(t - r)!}$$

$$\frac{h^r}{r!} Y^{(r)}_i(x_{i+1}) = \sum_{t=r}^{m} \frac{t!}{r!(t - r)!} \frac{h^t}{t!} Y^{(t)}_i(x_i)$$

$$= \sum_{t=r}^{m} \frac{t!}{r!(t - r)!} \sum_{j=1}^{N} \alpha^i_{tj} y_j.$$

Since $\frac{h^r}{r!} Y^{(r)}_{i+1} = \sum_{j=1}^{N} \alpha^{i+1}_{rj} y_j$ it follows that

$$(8.12)\qquad \alpha^{i+1}_{rj} = \sum_{t=r}^{m} \frac{t!}{r!(t - r)!} \alpha^i_{tj}.$$

If N is chosen to be an odd integer and if the ordinates through which each $Y_i(x)$ is required to pass are chosen in accordance with (7.3) then (8.12) is applicable for $i < \frac{1}{2}m - 2$ and for $i > N + 1 - \frac{1}{2}m$.

When the ordinates through which $Y_i(x)$ and $Y_{i+1}(x)$ are chosen in the same way, as for example when $Y_{i+1}(x)$ passes through y_{i-1}, y_i, and y_{i+1} and $Y_{i+1}(x)$ passes through y_i, y_{i+1}, and y_{i+2}, then clearly

$$\alpha_{rj}^{i+1} = \alpha_{r\,j-1}^{i}.$$

This formula is applicable, in accordance with (7.3), except near the end points and at the midpoint x_M, where $M = \frac{1}{2}(N + 1)$. If m is even we find that $Y_M(x) = Y_{M-1}(x)$ so that (8.12) applies. If m is odd then it follows from (7.3) that $Y_M(x) = Y_{M-2}(x)$ so that the induction formula (8.12) must be used twice. Noting that $\alpha_{r\,j+1}^{M-1} = \alpha_{rj}^{M-2}$ we have

$$\alpha_{rj}^{M} = \sum_{s=r}^{m} \frac{s!}{r!(s-r)!} \sum_{t=s}^{m} \frac{t!}{s!(t-s)!} \alpha_{t\,j+1}^{M-1}$$

when m is odd. Summarizing these results

$$\alpha_{rj}^{i+1} = \sum_{t=r}^{m} \frac{t!}{r!(t-r)!} \alpha_{tj}^{i} \qquad \text{for} \quad \begin{cases} i \leq \dfrac{m}{2} - 2 \\ i \geq N + 1 - \dfrac{m}{2} \\ i = M - 1, \text{ if } m \text{ is even} \end{cases}$$

$$\alpha_{rj}^{M} = \sum_{s=r}^{m} \frac{s!}{r!(s-r)!} \sum_{t=s}^{m} \frac{t!}{s!(t-s)!} \alpha_{t\,j+1}^{M-1} \text{ if } m \text{ is odd}$$

$$\alpha_{rj}^{i+1} = \alpha_{r\,j-1}^{i} \qquad \text{for} \quad \begin{cases} \dfrac{m}{2} - 2 < i < M - 1 \\ M - 2 < i < N + 1 - \dfrac{m}{2} \end{cases}$$

where, again, N is odd and $M = \frac{1}{2}(N + 1)$. These equations together with our previous results for determining the α^0_{rj} constitute a means for computing the coefficients α^i_{rj} for $i = 0, \cdots, N$; $r = 0, \cdots, m$; and $j = 1, \cdots, N$ for the boundary conditions $y(a) = y(b) = 0$ when N is chosen to be odd.

9. The Computation Procedure. Both Method I and Method II lead to algebraic problems of the type given by equation (7.13). In the former case the matrix D is diagonal and various simplifications occur. For both methods the computation proceeds in two similar steps: (1) computing the elements of the matrices L and D, and (2) solving equations (7.13) for λ and the y_j.

In Method I the matrix L is determined from the matrices a_{ij} given by equations (5.3) or (6.3) in accordance with equation (3.4).

In Method II step (1) begins with the computation of α^0_{rj}. We have initially a matrix with $m + 1$ columns, $r = 0, \cdots, m$ and $N + 1$ rows, $j = 0, \cdots, N$,

only the element (0,0) being different from 0. Then the transpose of the matrix $\|\alpha^0{}_{rj}\|$ is computed using the formulas derived in section 8. Thereafter the first row is no longer needed (assuming boundary conditions $y(a) = y(b) = 0$). Then we have routines which perform the following functions: (1) compute $\|\alpha^i{}_{rj}\|$ from $\|\alpha_{rj}^{i-1}\|$ using equations (8.13); (2) compute the matrix $\|P^i - Q^i\|$ using equations (7.8) and (7.9); (3) matrix multiplication $\sum_{r=0}^{m} \alpha^i{}_{rt}(P^i{}_{rt} - Q^i{}_{rt})$; (4) matrix multiplication $\sum_{i=0}^{m} \alpha^i{}_{tk}[\sum_{r=0}^{m} \alpha_{rj}(P^i{}_{rt} - Q^i{}_{rt})]$ and addition of the result to the previous partial sum; (5) compute the matrix $\|R^i\|$ using equations (7.10); (6) (master routine) step i and repeat until $i = N + 1$. Routines (3) and (4) are used to form the partial sums for both L and D. Likewise (3) and (5) may conveniently be combined into a single routine. Since $\|P^i - Q^i\|$, $\|R^i\|$ are both symmetric and can use the same storage locations, since L and D are symmetric, and since $\|\alpha^i\|^T \cdot \|P^i - Q^i\|$ and $\|\alpha^i\|^T \cdot \|R^i\|$ can use the same storage locations, a total of $\frac{1}{2}(m + 1)(m + 2) + N(N + 1) + 2N(m + 1)$ matrix storage locations are required. It is also time-saving to store the integers 0! through m! for use in computing the $\|\alpha^i\|$.

If the functions $q(x)$ and $r(x)$ are symmetric with respect to $x = \frac{1}{2}(a + b)$ it is time-saving to choose N to be odd and carry out the "integration" only from $x_M = \frac{1}{2}(a + b)$ to $x_{N+1} = b$. Then the correct result is obtained by replacing the elements $L_{jk}(D_{jk})$ and $L_{N-j+1,N-k+1}(D_{N-j+1,N-k+1})$ both by their sum for $j, k \leq M$. In obtaining the numerical results reported in this paper this procedure was followed. The matrix $\alpha^M{}_{rj}$ was computed utilizing a numerical differentiating routine which uses polynomial approximations, but a procedure similar to that outlined for obtaining $\alpha^0{}_{rj}$ could have been followed.

In both methods, the scheme used in step (2) for solving the equation

$$Ly = \lambda Dy$$

begins with the computation of the characteristic values and characteristic vectors of the matrix D, to obtain the decomposition $D = U^T \Delta U$, where Δ is a diagonal matrix containing the characteristic values of D and U is a unitary matrix containing its characteristic vectors. Then

$$[(\Delta^{-\frac{1}{2}}U^T)^T L(\Delta^{-\frac{1}{2}}U^T)](U\Delta^{\frac{1}{2}}Y) = \lambda(U\Delta^{\frac{1}{2}}Y).$$

The matrix shown in brackets is computed and then diagonalized to obtain the characteristic values λ and the vectors $U\Delta^{\frac{1}{2}}Y$. The characteristic vectors $Y = \Delta^{-\frac{1}{2}}U^T(U\Delta^{\frac{1}{2}}Y)$ are then computed. Both diagonalizations are done by the Jacobi method (see Gregory [3]). However for Method I advantage is taken of the fact that D is already a diagonal matrix.

Since, except for the factor rt in the definition of $P^i{}_{rt}$, the elements $P^i{}_{rt}$, $Q^i{}_{rt}$, $R^i{}_{rt}$ depend only on the sum $r + t$ and i, a total of $(N + 1)(6m + 2)$ quadratures, each over an interval of length h, are required. Computing the matrices L and D requires a total of $m(N + 1)(N^2 + N + 2m)$ multiplications. The most troublesome term in this expression is mN^3; this is to be compared with $50N^3$, which is approximately the number of multiplications required in the two diagonalizations. Thus the time required to compute the elements of L and D is small in comparison to the time required to solve the equation $Ly = \lambda Dy$.

10. Numerical Results and Their Discussion. In this section we shall report and discuss the results obtained by applying Methods I and II to the example given by equations (6.7) to (6.9). The results are given in tabular form. Each entry represents the value of $100(\lambda_k^* - \lambda_k)/\lambda_k$ where λ_k is the characteristic value as computed from equation (6.10) with $n = k$ and λ_k^* is the kth approximate characteristic value obtained by solving the equation

$$Ly = \lambda Dy.$$

Results on the comparison of the y_i which satisfy the above equation and $y(x_i)$ computed from equation (6.11) are also available but not reported herein because of space limitations.

TABLE 1. *$100(\lambda_k^* - \lambda_k)/\lambda_k$ from Method I with $N = 19$*

$k =$	1	2	3	4	5	6	7	8	9
1st Order	4.97	4.29	3.16	1.60	− .37	−2.75	−5.49	−8.59	−11.99
2nd Order	.00	− .03*	− .15	− .47*	−1.17	−2.46*	−4.60	−7.89	−12.60
2nd Order*	.00	.00*	− .14	− .47*	−1.17	−2.45*	−4.60	−7.88	−12.59

Not all the computed λ^*'s are reported in these tables. In case $N = 19$ we give nine λ^* and in case $N = 9$ we give five λ^*. The reason for this is that the true characteristic values λ_n for $n \geq 10$ when $N = 19$ and $n \geq 5$ when $N = 9$ correspond to functions which have a maximum within the first interval used. Hence any approximation of this function by the value of its ordinate at the end of the interval will not be a reasonable one and would be expected to lead to poor results, as was found to be the case.

In Table 1 which contains the results obtained by the use of Method I with $N = 19$ still an additional selection principle was used in choosing the value of λ_k^* reported. The first nine values of λ^* which were associated with values of y_i^*,

TABLE 2. *$100(\lambda_k^* - \lambda_k)/\lambda_k$ from Method II with $N = 9$*

m/k	1	2	3	4	5
2	.018	.309	1.249	4.192	7.498
3	.001	.052	.645	2.224	7.299
4	.000	.015	.022	.323	1.945
5	.000	.000	.019	.104	.767
6	.000	.000	.003	.106	.383
7	.000	.000	.000	.028	.723
8	.000	.000	.000	.004	.203

such that a continuous function passing through these ordinates would have the same number of zeros as the corresponding solution of the Sturm-Liouville equation, were used. That is, the known oscillation properties of solutions of Sturm-Liouville systems were used to distinguish between "true" approximations to λ_k and "spurious" ones when two close values of λ_k^* were obtained. The entries marked with an asterisk in Table 1 were one of a pair of computed values of λ^* which were numerically close.

The first line of Table 1 contains the results obtained by using the first order expressions for a_{ij} and w_i (equations (5.3) and (5.4) in equation (3.4)). The second line was obtained by using the second order expressions for a_{ij} and w_i (equations

(6.3) and (6.4)). The third line was obtained by modifying the even rows in the body of the matrix in equation (6.3) so that they are of the form

$$2^{-15} \quad -(\tfrac{1}{2} + 2^{-14}) \quad 0 \quad (\tfrac{1}{2} + 2^{-14}) \quad -2^{-15}$$

instead of

$$0 \quad -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \quad 0.$$

The fact that the entries of the third row are uniformly smaller than those of the second shows that there is no advantage to be obtained in Method I by requiring the a_{ij} and w_j to be chosen so that the resulting algebraic equation is a second order approximation to the original differential equation. Indeed this example shows that by modifying the a_{ij} and w_j satisfying this condition, so that the condition is violated, better results can be obtained.

Table 2 gives the results obtained by using Method II. The entries in a single row correspond to a fixed value of m labelling that row. As was pointed out earlier, the total computation time is not seriously changed as m is increased from 2 to 8. By using $m = 8$ one can obtain all the characteristic values one has a right to expect to get with an accuracy less than or equal to .203 per cent.

TABLE 3. *$100(\lambda_k^* - \lambda_k)/\lambda_k$ from Modified Method II (with $m = 2$)*

$k =$	1	2	3	4	5	6	7	8	9
$N = 19$	.00	−.01	− .05	− .13	− .32	−.67	−1.39	−3.09	−7.53
$N = 9$	−.010	−.137	− .655	−3.075	−18.943				
$N = 9$	.011	.126	.389	.674	1.34				
$N = 9$	.018	.308	1.249	4.192	7.498				

The first two rows in Table 3 contain results obtained by using a modification of Method II. In the first row of this table $N = 19$ and in the last three rows $N = 9$. The last row of Table 3 is the first row of Table 2. The modifications referred to are the following: The polynomials $Y_i(x)$ used to replace the functions $y(x)$ in the integral (2.2) are such that

$$\begin{aligned} Y_0(x) &= P_2(x) \\ Y_{i+1}(x_{i+1}) &= Y_i(x_{i+1}) \\ Y'_{i+1}(x_{i+1}) &= Y_i'(x_{i+1}). \end{aligned}$$

Further, for the first two rows of Table 3 the matrix D is made diagonal by evaluating

$$\int_a^b ry^2 dx$$

by means of Simpson's rule instead of replacing $y(x)$ by $Y(x)$ defined above.

This last approximation procedure means that the computation time required for solving the equation

$$Ly = \lambda Dy$$

is markedly decreased. However, it is no longer true that the λ_k^* are always greater than the λ_k.

On comparing the second and fourth rows of Table 3 we see that for the first four characteristic values the results obtained from the modified Method II are better than those given by the unmodified one for $N = 9$. Other computations have shown that this result does not hold for $m \neq 2$ if the class of admissible function is defined so as to have continuous derivatives at the points $x = x_i$ and if the matrix D is evaluated as in Method II.

In the third row of Table 3 the matrix D is evaluated as in Method II where the admissible functions are defined as for the first two rows.

The numerical results, although obtained only for a particular simple example, would seem to justify the use of the methods discussed above for the solution of more complicated Sturm-Liouville problems. They should be supplemented by a test for the accuracy of the results obtained for the characteristic values and characteristic functions applicable to problems for which the solution is not known. Various such tests are under investigation and will be reported on subsequently.

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On the Numerical Evaluation of the Stokes' Stream Function

1. Introduction. In the study of axially symmetric problems in fluid dynamics, the Stokes' stream function, that is, the function which satisfies

$$u_{xx} - \frac{1}{y} u_y + u_{yy} = 0, \quad y \neq 0, \tag{1.1}$$

is of considerable interest. This function is constant on the streamlines.

The problem to be considered is a Dirichlet type problem. Let G be a closed, bounded, simply connected plane region whose interior is denoted by R and whose boundary curve is denoted by S. Let G not contain any point where $y = 0$.

Let $g(x, y)$ be defined and continuous on S. The problem then is to produce a function $u(x, y)$ such that

a) $u(x, y) \equiv g(x, y)$, on S, and
b) $u(x, y)$ satisfies (1.1) in R.

Under general conditions, there exists a unique solution (Bernstein [1] and Courant and Hilbert [3]), and only such cases will be considered. However, the analytical determination of $u(x, y)$ is quite another story from that of its existence and usually offers what are at present insurmountable problems. The approach here, then, will be from a numerical analysis point of view.

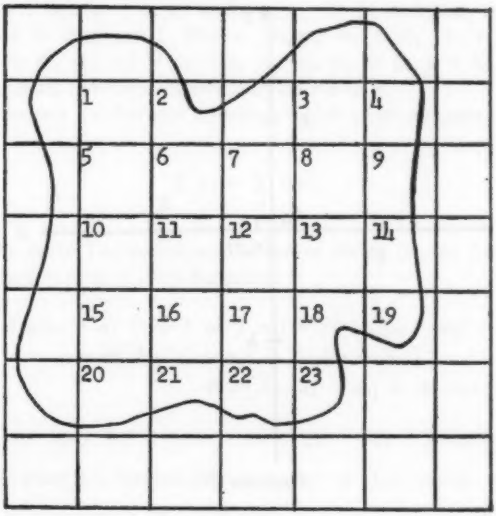

DIAGRAM 1

Let h be a fixed positive constant called the mesh size. Let (x_0, y_0) be an arbitrary, but fixed point of G and denote by G_h the set of all points of form $(x_0 + mh, y_0 + nh)$, contained in G, where m and n are integers. Two points (x_1, y_1) and (x_2, y_2) of G are called adjacent if and only if $(x_2 - x_1)^2 + (y_2 - y_1)^2 = h^2$. R_h, the interior of G_h, is the set of all points of G_h whose four adjacent points belong to G_h. S_h, the boundary of G_h, called the lattice boundary, is defined by $S_h = G_h - R_h$. It is also assumed that any pair of points of R_h can be joined by a connected polygonal arc consisting of straight line segments which join adjacent points of R_h.

The technique proposed will involve the application of a difference equation to yield a system of linear equations whose solution is an approximation to the solution of the formulated Dirichlet problem at the points of G_h.

The question of solving the linear system will not be considered. Systems with unique solutions may be approached by means of Cramer's rule, Gauss' Elimina-

tion Procedure, matrix inversion, relaxation, iteration, gradient methods, and other techniques (Geiringer [4], Greenspan [6], Milne [7], Shortley et al [8], NBS [9], Young [13, 14]).

The author is indebted to D. M. Young for valuable criticisms and suggestions.

2. General Method. Suppose G_h consists of n points. Number these points in a one-to-one fashion with the integers $1, 2, 3, \cdots, n$. Denote the coordinates of the point numbered k by (x_k, y_k) and the unknown stream function at (x_k, y_k) by $u(x_k, y_k) \equiv u_k$, for $k = 1, 2, \cdots, n$.

Let (x_i, y_i) be an arbitrary point of S_h, the lattice boundary. Approximate u_i by $g(x', y')$, where (x', y') is the nearest point of S to (x_i, y_i). If (x', y') is not unique, choose any one of the set of nearest points and use it. The problem of

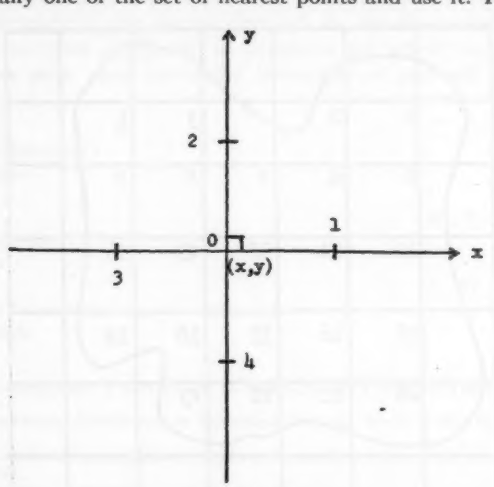

DIAGRAM 2

finding numerical approximations to $u(x, y)$ on the lattice boundary is, though crudely done, adequate for present purposes. For diagram 1 this means that at the points 1, 2, 3, 4, 5, 7, 9, 10, 14, 15, 19, 20, 21, 22, 23 the values of the stream function have been approximated by values at nearby points of S.

We then require that at each point (x_i, y_i) of R_h, the function u satisfy

$$(2.1)\quad 4u(x_i, y_i) - u(x_i + h, y_i) - \left(1 - \frac{h}{2y_i}\right) u(x_i, y_i + h) - u(x_i - h, y_i) - \left(1 + \frac{h}{2y_i}\right) u(x_i, y_i - h) = 0.$$

It is convenient, in practice, to use the subscript notation. For example, for diagram 1, if $i = 12$, (2.1) becomes

$$4u_{12} - u_{13} - \left(1 - \frac{h}{2y_{12}}\right) u_7 - u_{11} - \left(1 + \frac{h}{2y_{12}}\right) u_{17} = 0.$$

Note that u_{12}, u_{13}, u_{11}, u_{17} are unknowns while u_7 is a constant determined by the method for points of S_h. Application of (2.1) to each point of R_h thereby results in a system of linear equations which, when solved, yields the remaining numerical approximations.

Equation (2.1) is only one of a variety of possible difference equations which can be developed, as described in the next section.

3. Development of a General Difference Equation. We shall seek a solution $u(x, y)$ which is of class C^4 and use the notation $\partial^{i+j}u/\partial x^i \partial y^j \equiv u_{i,j}$, $i + j \leq 4$, and $A_{ij} \equiv u_{i,j}/i!j!$.

Let (x, y) be an arbitrary point of R_h. The aim will be to develop a "5-point formula," so consider here only (x, y) and its four adjacent points. Let (x, y), $(x + h, y)$, $(x, y + h)$, $(x - h, y)$, and $(x, y - h)$ be denoted, respectively, by 0, 1, 2, 3, 4, as in diagram 2. Hence $(x_0, y_0) = (x, y)$, $(x_1, y_1) = (x + h, y)$, $(x_2, y_2) = (x, y + h)$, $(x_3, y_3) = (x - h, y)$, $(x_4, y_4) = (x, y - h)$. Also let $O(h^n)$ represent any and all functions of order at least n in h.

In order to deduce a difference equation which approximates (1.1), let

$$L(u) = \sum_0^4 \alpha_i u_i. \tag{3.1}$$

By use of the finite Taylor series expansions about (x_0, y_0) for the various u_i and by the definition of A_{ij}, (3.1) becomes

$$\begin{aligned} L(u) = {} & A_{00}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + h[A_{10}(\alpha_1 - \alpha_3) + A_{01}(\alpha_2 - \alpha_4)] \\ & + h^2[A_{20}(\alpha_1 + \alpha_3) + A_{02}(\alpha_2 + \alpha_4)] \\ & + h^3[A_{30}(\alpha_1 - \alpha_3) + A_{03}(\alpha_2 - \alpha_4)] + O.(h^4) \end{aligned} \tag{3.2}$$

Equation (1.1) may be rewritten as $yu_{2,0} - u_{0,1} + yu_{0,2} = 0$. Since u is assumed to be of class C^4, further differentiation of (1.1) yields

$$yu_{2,0} - u_{1,1} + yu_{1,2} = 0$$
$$u_{2,0} + yu_{2,1} + yu_{0,3} = 0.$$

Rewriting these in terms of the A_{ij}'s and solving, one finds

$$A_{20} = \frac{A_{01}}{2y} - A_{02};\quad A_{30} = \frac{A_{11}}{6y} - \frac{A_{12}}{3};\quad A_{21} = -3A_{03} + \frac{A_{02}}{y} - \frac{A_{01}}{2y^2}. \tag{3.3}$$

Substitution of (3.3) into (3.2) yields

$$\begin{aligned} L(u) = {} & A_{00}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\ & + h\left[A_{01}\left(\frac{\alpha_1 h}{2y} + \frac{\alpha_3 h}{2y} + \alpha_2 - \alpha_4\right) + A_{10}(\alpha_1 - \alpha_3)\right] \\ & + h^2[A_{02}(\alpha_2 + \alpha_4 - \alpha_1 - \alpha_3)] \\ & + h^3[A_{30}(\alpha_1 - \alpha_3) + A_{03}(\alpha_2 - \alpha_4)] + O(h^4). \end{aligned} \tag{3.4}$$

Let

$$
(3.5) \qquad
\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= \epsilon_1 \\
\frac{\alpha_1 h}{2y} + \frac{\alpha_3 h}{2y} + \alpha_2 - \alpha_4 &= \epsilon_2 \\
\alpha_1 - \alpha_3 &= \epsilon_3 \\
\alpha_2 + \alpha_4 - \alpha_1 - \alpha_3 &= \epsilon_4 \\
\alpha_2 - \alpha_4 &= \epsilon_5,
\end{aligned}
$$

where

$$\epsilon_1 = O(h^4), \quad \epsilon_2 = O(h^3), \quad \epsilon_3 = O(h^3), \quad \epsilon_4 = O(h^2), \quad \epsilon_5 = O(h).$$

The solution to system (3.5) is

$$
(3.6) \qquad
\begin{aligned}
\alpha_1 &= \frac{\epsilon_2 y}{h} - \frac{\epsilon_5 y}{h} + \frac{\epsilon_3}{2} & \alpha_2 &= \frac{\epsilon_2 y}{h} + \frac{\epsilon_4}{2} + \frac{\epsilon_5}{2} - \frac{\epsilon_5 y}{h} \\
\alpha_3 &= \frac{\epsilon_2 y}{h} - \frac{\epsilon_5 y}{h} - \frac{\epsilon_3}{2} & \alpha_4 &= \frac{\epsilon_2 y}{h} + \frac{\epsilon_4}{2} - \frac{\epsilon_5}{2} - \frac{\epsilon_5 y}{h}
\end{aligned}
$$

$$\alpha_0 = \epsilon_1 - \epsilon_4 + 4\frac{\epsilon_5 y}{h} - 4\frac{\epsilon_2 y}{h}.$$

Finally, setting (3.1) equal to zero and substituting (3.6) in (3.1) yield the general difference equation (3.7) which approximates the differential equation (1.1)

$$
(3.7) \qquad
\begin{aligned}
&\left(\epsilon_1 - \epsilon_4 + \frac{4\epsilon_5 y}{h} - \frac{4\epsilon_2 y}{h}\right) u_0 + \left(\frac{\epsilon_2 y}{h} - \frac{\epsilon_5 y}{h} + \frac{\epsilon_3}{2}\right) u_1 \\
&\quad + \left(\frac{\epsilon_2 y}{h} + \frac{\epsilon_4}{2} + \frac{\epsilon_5}{2} - \frac{\epsilon_5 y}{h}\right) u_2 + \left(\frac{\epsilon_2 y}{h} - \frac{\epsilon_5 y}{h} - \frac{\epsilon_3}{2}\right) u_3 \\
&\quad + \left(\frac{\epsilon_2 y}{h} + \frac{\epsilon_4}{2} - \frac{\epsilon_5}{2} - \frac{\epsilon_5 y}{h}\right) u_4 = 0.
\end{aligned}
$$

The process, then, was to find α_i such that: $L(u) + O(h^4) \equiv 0$, where u satisfies (1.1), and to then use $L(u) = 0$ for the general approximation.

Equation (2.1), which will be used throughout, is deduced from (3.7) by letting $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$, $\epsilon_5 = h$, by dividing through by y, and by noting $u_0 = u(x, y)$, $u_1 = u(x + h, y)$, $u_2 = u(x, y + h)$, $u_3 = u(x - h, y)$, $u_4 = u(x, y - h)$.

Other difference equations may be constructed by selecting the ϵ_i's differently. One of these of particular interest is one which would yield a linear system which has a symmetric coefficient matrix. This may be done in the following manner. Number the points as in diagram 1 so that the numbers increase from left to right and also increase if one reads down any column of points. Apply difference equation

$$
(3.8) \qquad
\begin{aligned}
&u_0\left(+\frac{2}{y} + \frac{2}{2y + h} + \frac{2}{2y - h}\right) - \frac{u_1}{y} - \frac{u_3}{y} \\
&\quad - u_2\left(\frac{2}{2y + h}\right) - u_4\left(\frac{2}{2y - h}\right) = 0, \quad |2y| \neq h
\end{aligned}
$$

to the points of R_h. Start with the first, or highest, row of points; traverse the row from left to right; proceed to the next row; traverse the row from left to right; proceed to the next row, etc.; continue the process until all the points of R_h have been used. Equation (3.8) may be constructed by a numerical procedure for certain self-adjoint elliptic differential equations (Young [15]), or directly from (3.7) by letting

$$\epsilon_1 = 0, \quad \epsilon_2 = \frac{-h}{y^2} + \frac{4h}{(2y+h)(2y-h)}, \quad \epsilon_3 = 0,$$

$$\epsilon_4 = \frac{-8y}{(2y+h)(2y-h)} + \frac{2}{y}, \quad \epsilon_5 = \frac{+4h}{(2y+h)(2y-h)}.$$

4. A Theorem on an Error Bound. In this section, let $u(x, y)$ be the solution of the Dirichlet problem being considered and $U(x, y)$ the solution of the numerical method described in section 2. This means that

$$-4U_0 + U_1 + U_3 + \left(1 - \frac{h}{2y}\right) U_2 + \left(1 + \frac{h}{2y}\right) U_4 = 0 \tag{4.1}$$

for any arrangement of points of G_h like that of diagram 2.

LEMMA 1. *The solution of the system of linear equations which results by application of the process of section 2 with $y \neq 0$ and $\left|\frac{h}{y}\right| < 1$ is unique.*

Proof. It is sufficient to show that the determinant of the system of linear equations is not zero and this is done by demonstrating that the only solution of the homogeneous system which results by considering $g(x, y) \equiv 0$ on S is the zero vector. Suppose then there exists a nontrivial solution for the homogeneous system. For some point of R_h, $U \neq 0$. Suppose $U > 0$. Let the largest value M occur at (x_0, y_0). Then

$$4U_0 = U_1 + U_3 + \left(1 - \frac{h}{2y}\right) U_2 + \left(1 + \frac{h}{2y}\right) U_4, \tag{4.2}$$

and

$$M = U_0 \geq U_i, \quad i = 1, 2, 3, 4. \tag{4.3}$$

Case 1. $U_2 = U_4$. Then (4.1) becomes

$$4U_0 = U_1 + U_2 + U_3 + U_4. \tag{4.4}$$

But (4.3) and (4.4) imply $U_0 = U_1 = U_2 = U_3 = U_4 = M$. Hence one can apply the same argument at (x_1, y_1) and continue in a finite number of steps to show that U at a boundary point is equal to M. This is a contradiction since $g(x, y) \equiv 0$ on S and the method described for points of S_h implies U at all lattice boundary points is zero.

Case 2. $U_2 > U_4$. Let $U_4 = U_2 - k$, $k > 0$. Then

$$4U_0 = U_1 + 2U_2 + U_3 - k\left(1 + \frac{h}{2y}\right),$$

and

$$4U_0 < U_1 + 2U_2 + U_3, \quad \text{since} \quad 1 + \frac{h}{2y} > 0.$$

But this is impossible since $U_1 \leq U_0$, $U_2 \leq U_0$, $U_3 \leq U_0$.

Case 3. $U_4 > U_2$. The discussion is analogous to that for Case 2 and leads to a contradiction.

Hence, in all three cases, the assumption that $U > 0$ yields a contradiction. Contradictions are similarly reached if $U < 0$ is assumed. Hence the only solution is the trivial solution. This completes the proof.

Using the same ideas and techniques first set forth by Gershgorin [5], we apply these by assuming in all that follows that $\sigma = h/2y$, $|\sigma| < 1$ and by defining

$$\lambda[u(x, y)] \equiv h^{-2}\left\{-4u(x, y) + u(x + h, y) + u(x - h, y) + \left(1 - \frac{h}{2y}\right)u(x, y + h) + \left(1 + \frac{h}{2y}\right)u(x, y - h)\right\}. \tag{4.5}$$

LEMMA 2. *If* $\lambda[v] \leq 0$ *on* R_h *and* $v \geq 0$ *on* S_h, *then* $v \geq 0$ *on* R_h.

Proof. Suppose $v < 0$ for some point of R_h. Then there exists a point (x_0, y_0) of R_h where v is a minimum. Hence

$$v(x_0, y_0) < 0, \text{ and, } v(x_0, y_0) \leq v(x, y), \text{ for all } (x, y) \text{ of } R_h. \tag{4.6}$$

Now, $\lambda[v] \leq 0$, by assumption, so that, equivalently

$$v_0 \geq \frac{1}{4}\left[v_1 + \left(1 - \frac{h}{2y}\right)v_2 + v_3 + \left(1 + \frac{h}{2y}\right)v_4\right]. \tag{4.7}$$

But (4.6) and (4.7) imply, in a manner similar to that used in Lemma 1, that $v_0 = v_1 = v_2 = v_3 = v_4$. In a finite number of steps this leads to the result that $v < 0$ for some point of S_h, which is a contradiction. This completes the proof.

LEMMA 3. *If* $-|\lambda[v_1]| \geq \lambda[v_2]$ *in* R_h *and* $|v_1| \leq v_2$ *on* S_h, *then* $|v_1| \leq v_2$ *on* R_h.

Proof. $v_2 - v_1 \geq 0$ on S_h and $\lambda[v_2 - v_1] = \lambda[v_2] - \lambda[v_1]$. Since $\lambda[v_2] + |\lambda[v_1]| \leq 0$, then $\lambda[v_2] - \lambda[v_1] \leq \lambda[v_2] + |\lambda[v_1]| \leq 0$. Hence $\lambda[v_2 - v_1] \leq 0$. By Lemma 2, $v_2 - v_1 \geq 0$ on R_h.

Also, $v_2 + v_1 \geq v_2 - |v_1| \geq 0$ on S_h. Then, $\lambda[v_2 + v_1] = \lambda[v_2] + \lambda[v_1] \leq 0$. Hence $v_2 + v_1 \geq 0$ on R_h. But $v_2 + v_1 \geq 0$ on R_h and $v_2 - v_1 \geq 0$ on R_h imply $v_2 \geq |v_1|$ on R_h. This completes the proof.

LEMMA 4. *If* $|\lambda[v]| \leq A$ *on* R_h *and if* $|v| \leq B$ *on* S_h, *and* r *is the radius of a properly selected circle, as described below, which contains* G, *then* $|v| \leq \frac{A}{4}r^2 + B$.

Proof. Let

$$w(x, y) = \left[\frac{Ar^2}{4}\left\{1 - \frac{(x - x_0)^2 + (y - y_0)^2}{r^2}\right\} + B\right],$$

where $(x - x_0)^2 + (y - y_0)^2 = r^2$ is the equation of any circle containing G for which $y_0/y \geq 1$, for all y in G. At least one such circle exists since G is simply connected, bounded, closed and $y \neq 0$, which imply that y is of constant sign.

Direct calculation yields $\lambda[w] = -\frac{A}{2}(1 + y_0/y) \leq -A$. Also, $w \geq B$ on S_h.

Now, since $|\lambda[v]| \leq A$ on R_h, by assumption, and it has been shown that $\lambda[w] \leq -A$, or equivalently, $-\lambda[w] \geq A$, it follows that $-\lambda[w] \geq -A \geq \lambda[v]$ on R_h. Since $|v| \leq B$ on S_h, and $w \geq B$ on S_h, $w \geq |v|$. Hence, $-|\lambda[v]| \geq \lambda[w]$ on R_h and $w \geq |v|$ on S_h. By Lemma 3, $w \geq |v|$ on R_h, or $|v| \leq w \leq \frac{Ar^2}{4} + B$.

This completes the proof.

THEOREM. *If $u(x, y)$ is of class C^5 in a closed region G, $y \neq 0$ in G, u denotes the solution of the stream equation, U denotes the solution of the linear system which results from application of* (2.1) *in the method of section* 2, $|\sigma| = \left|\frac{h}{y}\right| < 1$, *then*

$$|U - u| \leq \frac{r^2}{4}\left[\frac{h^2M_3}{3!\bar{y}} + \frac{h^2M_4}{3!} + \frac{h^4M_5}{2\cdot 4!\bar{y}}\right] + 2hM_1,$$

where

$$M_1 = \operatorname{Max}\left[\operatorname*{Max}_{(x,y)\epsilon G}|u_{1,0}|, \operatorname*{Max}_{(x,y)\epsilon G}|u_{0,1}|\right], \quad M_3 = \operatorname*{Max}_{(x,y)\epsilon G}|u_{0,3}|,$$

$$M_4 = \operatorname{Max}\left[\operatorname*{Max}_{(x,y)\epsilon G}|u_{0,4}|, \operatorname*{Max}_{(x,y)\epsilon G}|u_{4,0}|\right], \quad M_5 = \operatorname*{Max}_{(x,y)\epsilon G}|u_{0,5}|,$$

$\bar{y} = GLB|y|$ in G, r = radius of any circle of type described in Lemma 4.

Proof. Let $R = \lambda[u] - \left(u_{xx} - \frac{1}{y}u_y + u_{yy}\right)$. Substitution of the finite Taylor series expansions for u_1, u_2, u_3, u_4 in $\lambda[u]$ and use of the Mean Value Theorem yields

$$R = \frac{h^2}{4!}[u_{4,0}(\rho_1, y) + u_{0,4}(x, \rho_2) + u_{4,0}(\rho_3, y) + u_{0,4}(x, \rho_4)] - \frac{h^2}{3!y}u_{0,3}(x, y) + \frac{h^4}{2\cdot 4!y}[u_{0,5}(x, \rho_5)],$$

where $\rho_2 < \rho_5 < \rho_4$. Hence

$$|R| \leq \frac{M_4h^2}{3!} + \frac{M_3h^2}{3!\bar{y}} + \frac{M_5h^4}{2\cdot 4!\bar{y}}.$$

Since $u_{xx} - \frac{1}{y}u_y + u_{yy} = 0$,

$$\left|\lambda[u] - \left(u_{xx} - \frac{1}{y}u_y + u_{yy}\right)\right| = |\lambda[u]| = |R| \leq \frac{h^2M_3}{3!\bar{y}} + \frac{h^2M_4}{3!} + \frac{h^4M_5}{2\cdot 4!\bar{y}}. \tag{4.8}$$

Also, for any point of S_h, U was taken as the value of $g(x', y')$ at the nearest point (x', y') on the boundary S, and $g(x', y') = u(x', y')$ on S. Thereby, for any point (x, y) of S_h,

$$(4.9) \quad |U(x, y) - u(x, y)| = |g(x', y') - u(x, y)| = |u(x', y') - u(x, y)|,$$

where (x', y') is a point of S and $(x - x')^2 + (y - y')^2 \leq 2h^2$. Therefore

$$(4.10) \quad \begin{aligned} |U(x, y) - u(x, y)| &= |u(x', y') - u(x, y') + u(x, y') - u(x, y)| \\ &\leq |u_x(\rho_6, y')(x - x')| + |u_y(x, \rho_7)(y - y')| \\ &\leq M_1[|y' - y| + |x' - x|] \leq 2hM_1. \end{aligned}$$

It must also be noted that $\lambda[U] = 0$, by (4.1). Hence

$$(4.11) \quad |\lambda[u - U]| = |\lambda[u] - \lambda[U]| = |\lambda[u]|.$$

Applying Lemma 4 to (4.8), (4.10), and (4.11), one finds that on R_h

$$(4.12) \quad |U - u| \leq \frac{r^2}{4}\left\{\frac{h^2M_3}{3!g} + \frac{h^2M_4}{3!} + \frac{h^4M_6}{2\cdot 4!g}\right\} + 2hM_1.$$

This completes the proof.

COROLLARY. *Under the assumptions of the theorem just proved, $U \to u$, as $h \to 0$.*

Under general conditions set forth by Collatz [2], the term $2hM_1$ in the error bound for $|U - u|$ may be replaced by a term of type $O(h^2)$ by replacing the unesthetic method for deciding U on S_h as follows. Let (x_1, y_1) be a point of S, (x_2, y_2) of R_h, and (x_0, y_0) of S_h. Let PQ (diagram 3) be parallel to a coordinate axis and let δ be the absolute value of the distance between (x_0, y_0) and (x_1, y_1). Choose U_0 by

$$U_0 = \frac{hg(x_1, y_1) + \delta U_2}{h + \delta}.$$

5. Use of Diagonal Points. After a numerical solution has been found on a given set G_h, solutions at other points may be approximated rapidly as follows. Let 5, 6, 7, 8 represent four points of G_h which are vertices of a square. (See

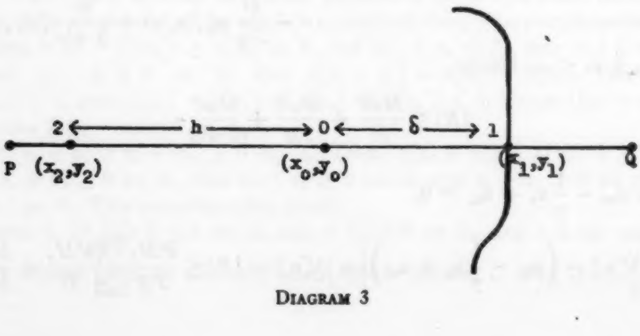

DIAGRAM 3

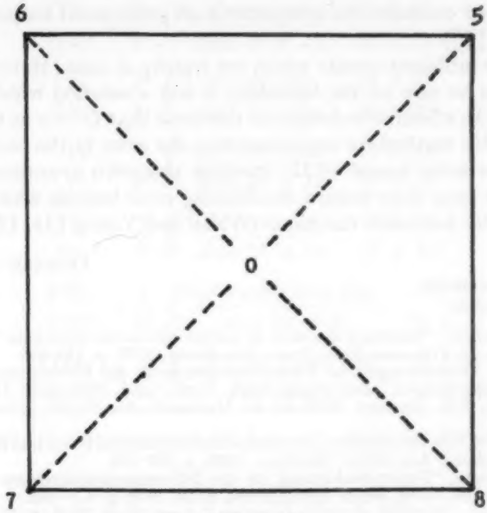

DIAGRAM 4

diagram 4.) Let 0 represent the diagonal point of the square. Use

$$(5.1) \quad U_0 = \frac{(2-\sigma-\sigma^2)(U_5+U_6)+(2+\sigma-\sigma^2)(U_7+U_8)}{4(2-\sigma^2)}, \quad \sigma = \frac{h}{y_0}.$$

The unexciting calculations will here be replaced by a general outline of how to establish (5.1). Let $L(u) = \alpha_0 u_0 + \alpha_5 u_5 + \alpha_6 u_6 + \alpha_7 u_7 + \alpha_8 u_8$. Expand u_5, u_6, u_7, u_8 in Taylor series about (x_0, y_0) and substitute these expressions into $L(u)$. Simplify by replacing $u_{i,j}$ by $i!j!A_{ij}$. Apply identities (4.3) and combine as in (4.4). Set the coefficients equal to ϵ_i's. This system of equations is

$$\alpha_0+\alpha_5+\alpha_6+\alpha_7+\alpha_8=\epsilon_1, \quad \alpha_5-\alpha_6-\alpha_7+\alpha_8=\epsilon_2, \quad \alpha_5+\alpha_6-\alpha_7-\alpha_8=\epsilon_5,$$

$$\alpha_5\left(1+\frac{\sigma}{2}-\frac{\sigma^2}{2}\right)+\alpha_6\left(1+\frac{\sigma}{2}-\frac{\sigma^2}{2}\right) + \alpha_7\left(-1+\frac{\sigma}{2}+\frac{\sigma^2}{2}\right)+\alpha_8\left(-1+\frac{\sigma}{2}+\frac{\sigma^2}{2}\right)=\epsilon_3,$$

$$\alpha_5\left(1+\frac{\sigma}{6}\right)-\alpha_6\left(1+\frac{\sigma}{6}\right)+\alpha_7\left(1-\frac{\sigma}{6}\right)-\alpha_8\left(1-\frac{\sigma}{6}\right)=\epsilon_4,$$

where $\epsilon_1 = O(h^4)$, $\epsilon_2 = O(h^3)$, $\epsilon_3 = O(h^3)$, $\epsilon_4 = O(h^2)$, $\epsilon_5 = O(h)$. A general formula can be established, but by letting $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$, $\epsilon_5 = h$, solving for the α_i's and using the method of section 3, one deduces (5.1).

6. Concluding Remarks. The method and ideas developed can be used in an analogous fashion to effect the numerical solution of problems involving any one of the class of equations

$$(6.1) \qquad u_{xx} + \frac{k}{y} u_y + u_{yy} = 0,$$

where k is a fixed constant. Such equations are confronted frequently in physics (Shortley et al [8]).

Finally, two problems remain which are worthy of note. Many physical situations allow y to be zero on the boundary S and a solution which satisfies (6.1) in R is desired. An effective technique in the sense that $U \to u$ as $h \to 0$ is needed. Also, a reasonable method for approximating the error in the numerical solution is wanting. The error bound (4.12) involves unknown quantities and the only work which has been done toward establishing error bounds which can be calculated has been for harmonic functions (Walsh and Young [11, 12]).

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Tables of Values of 16 Integrals of Algebraic-Hyperbolic Type

This paper gives tables of values of the following 16 integrals of algebraic-hyperbolic type.

$$(1)\qquad \begin{matrix} I_k \\ I_k^* \end{matrix} = \frac{2^k}{k!}\int_0^\infty \frac{x^k dx}{\sinh 2x \pm 2x}, \quad \begin{matrix} (k \geqslant 1) \\ (k \geqslant 3) \end{matrix}$$

$$(2)\qquad \begin{matrix} II_k \\ II_k^* \end{matrix} = \frac{2^k}{k!}\int_0^\infty \frac{x^k e^{-2x} dx}{\sinh 2x \pm 2x}, \quad \begin{matrix} (k \geqslant 1) \\ (k \geqslant 3) \end{matrix}$$

$$\text{(3)}\qquad \begin{matrix} III_k \\ III_k^* \end{matrix} = \frac{2^k}{k!}\int_0^\infty \frac{x^k \tanh x dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqslant 0) \\ (k \geqslant 2) \end{matrix}$$

$$\text{(4)}\qquad \begin{matrix} IV_k \\ IV_k^* \end{matrix} = \frac{2^k}{k!}\int_0^\infty \frac{x^k \coth x dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqslant 2) \\ (k \geqslant 4) \end{matrix}$$

$$\text{(5)}\qquad \begin{matrix} V_k \\ V_k^* \end{matrix} = \frac{1}{k!}\int_0^\infty \frac{x^k \sinh x dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqslant 0) \\ (k \geqslant 2) \end{matrix}$$

$$\text{(6)}\qquad \begin{matrix} VI_k \\ VI_k^* \end{matrix} = \frac{1}{k!}\int_0^\infty \frac{x^k \cosh x dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqslant 1) \\ (k \geqslant 3) \end{matrix}$$

$$\text{(7)}\qquad \begin{matrix} VII_k \\ VII_k^* \end{matrix} = \frac{1}{k!}\int_0^\infty \frac{x^k \tanh x \sinh x dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqslant 0) \\ (k \geqslant 1) \end{matrix}$$

$$\text{(8)}\qquad \begin{matrix} VIII_k \\ VIII_k^* \end{matrix} = \frac{1}{k!}\int_0^\infty \frac{x^k \coth x \cosh x dx}{\sinh 2x \pm 2x}. \qquad \begin{matrix} (k \geqslant 2) \\ (k \geqslant 4) \end{matrix}$$

The integral (7) also converges for $k = -1$. It can be shown that

$$\int_0^\infty \frac{\tanh x \sinh x dx}{x(\sinh 2x + 2x)} = \ln 3 - \sum_{n=0}^{\infty} \frac{1 - III_{2n}}{(2n+1)2^{2n}} = 0.36236\ 7.$$

With the factors as shown, the integrals tend to unity as the integer k tends to infinity, except II_k and II_k^* both of which tend asymptotically to $2^{-(k+1)}$. The first four integrals, which are called Howland's integrals, arise from the problem of a symmetrically perforated strip or a notched strip (Howland [1],

TABLE 1

k	I_k	II_k	III_k	IV_k	V_k	VI_k	VII_k	$VIII_k$
0	—	—	0.29662 0	—	0.52685 6	—	0.36980 1	—
1	0.76857 5	0.22012 0	0.47443 0	—	0.73884 4	0.91747 7	0.65893 5	—
2	0.76784 7	0.08792 7	0.63084 1	1.09596 7	0.86986 6	0.92259 0	0.83403 0	1.02797 8
3	0.82771 0	0.04334 8	0.75383 6	0.94313 8	0.94005 4	0.95831 1	0.92528 4	0.98312 4
4	0.88350 7	0.02258 3	0.84283 9	0.93632 3	0.97401 0	0.98049 8	0.96827 5	0.98810 2
5	0.92547 6	0.01192 3	0.90323 7	0.95164 4	0.98924 9	0.99154 5	0.98711 6	0.99405 4
6	0.95419 2	0.00628 8	0.94219 2	0.96751 5	0.99571 1	0.99651 5	0.99494 2	0.99736 1
7	0.97269 9	0.00329 5	0.96631 5	0.97953 8	0.99833 7	0.99861 5	0.99806 6	0.99890 2
8	0.98412 4	0.00171 3	0.98077 1	0.98763 5	0.99936 9	0.99946 4	0.99927 5	0.99956 1
9	0.99094 9	0.00088 4	0.98920 7	0.99274 6	0.99976 5	0.99979 7	0.99973 2	0.99983 0
10	0.99492 2	0.00045 3	0.99402 5	0.99583 9	0.99991 3	0.99992 4	0.99990 2	0.99993 5
11	0.99718 9	0.00023 1	0.99673 0	0.99765 4	0.99996 8	0.99997 2	0.99996 5	0.99997 6
12	0.99846 0	0.00011 7	0.99822 7	0.99869 5	0.99998 9	0.99999 0	0.99998 7	0.99999 1
13	0.99916 4	0.00005 9	0.99904 6	0.99928 3	0.99999 6	0.99999 6	0.99999 5	0.99999 7
14	0.99954 9	0.00003 0	0.99949 0	0.99960 9	0.99999 9	0.99999 9	0.99999 8	0.99999 9
15	0.99975 9	0.00001 5	0.99972 9	0.99978 9	0.99999 9	1.00000 0	0.99999 9	1.00000 0
16	0.99987 1	0.00000 8	0.99985 6	0.99988 6	1.00000 0		1.00000 0	
17	0.99993 2	0.00000 4	0.99992 4	0.99993 9				
18	0.99996 4	0.00000 2	0.99996 0	0.99996 8				
19	0.99998 1	0.00000 1	0.99997 9	0.99998 3				
20	0.99999 0	0.00000 0	0.99998 9	0.99999 1				
21	0.99999 5		0.99999 4	0.99999 5				
22	0.99999 7		0.99999 7	0.99999 7				
23	0.99999 9		0.99999 8	0.99999 9				
24	0.99999 9		0.99999 9	0.99999 9				
25	1.00000 0		1.00000 0	1.00000 0				

TABLE 2

k	I_k^*	II_k^*	III_k^*	IV_k^*	V_k^*	VI_k^*	VII_k^*	$VIII_k^*$
1	—	—	—	—	—	—	1.51115 5	—
2	—	—	2.13561 8	—	1.40879 6	—	1.15546 4	—
3	2.03871 1	0.46071 4	1.41506 3	—	1.10522 6	1.20400 4	1.05738 7	—
4	1.35329 4	0.09931 6	1.19555 3	1.70756 9	1.03438 8	1.05121 3	1.02214 3	1.08171 9
5	1.15686 4	0.03241 3	1.10049 3	1.24000 1	1.01212 3	1.01624 8	1.00860 2	1.02146 1
6	1.07673 0	0.01261 7	1.05358 0	1.10538 5	1.00439 8	1.00556 4	1.00332 5	1.00686 4
7	1.03925 1	0.00539 1	1.02903 4	1.05082 2	1.00161 1	1.00196 5	1.00127 3	1.00233 9
8	1.02053 8	0.00243 3	1.01583 4	1.02560 8	1.00059 1	1.00070 3	1.00048 3	1.00081 7
9	1.01087 0	0.00113 6	1.00864 7	1.01319 9	1.00021 6	1.00025 2	1.00018 1	1.00028 9
10	1.00578 5	0.00054 2	1.00471 5	1.00688 6	1.00007 9	1.00009 1	1.00006 7	1.00010 2
11	1.00308 5	0.00026 3	1.00256 4	1.00361 5	1.00002 9	1.00003 2	1.00002 5	1.00003 6
12	1.00164 5	0.00012 8	1.00139 0	1.00190 3	1.00001 0	1.00001 2	1.00000 9	1.00001 3
13	1.00087 6	0.00006 3	1.00075 0	1.00100 3	1.00000 4	1.00000 4	1.00000 3	1.00000 5
14	1.00046 6	0.00003 1	1.00040 3	1.00052 9	1.00000 1	1.00000 1	1.00000 1	1.00000 2
15	1.00024 7	0.00001 6	1.00021 6	1.00027 8	1.00000 0	1.00000 1	1.00000 0	1.00000 1
16	1.00013 1	0.00000 8	1.00011 5	1.00014 6		1.00000 0		1.00000 0
17	1.00006 9	0.00000 4	1.00006 1	1.00007 7				
18	1.00003 6	0.00000 2	1.00003 3	1.00004 0				
19	1.00001 9	0.00000 1	1.00001 7	1.00002 1				
20	1.00001 0	0.00000 0	1.00000 9	1.00001 1				
21	1.00000 5		1.00000 5	1.00000 6				
22	1.00000 3		1.00000 3	1.00000 3				
23	1.00000 1		1.00000 1	1.00000 2				
24	1.00000 1		1.00000 1	1.00000 1				
25	1.00000 0		1.00000 0	1.00000 0				

Howland and Stevenson [2], Ling [3, 4]). The remaining twelve integrals, save VII_k and $VIII_k^*$, arise from the problem of an unsymmetrically perforated strip (Ling [5]).

The four Howland integrals I_k, I_k^*, II_k, II_k^* were tabulated by Howland [1] and Howland and Stevenson [2] to 5D. In a recent paper by Nelson and the present writer [6] these integrals were recalculated and tabulated to 6D. The values of I_k when k is odd were also tabulated by Nelson [7] to 9D. (However our I_k is designated by Nelson as a_k. Nelson has also calculated the remaining Howland integrals to 9D, but this work is unpublished.) For convenience of reference, values of the four Howland integrals are reproduced in Tables 1 and 2. It will be shown that the remaining twelve integrals can be evaluated in terms of the four Howland integrals.

The four integrals III_k, III_k^*, IV_k, IV_k^* may be evaluated by splitting the integrands and then integrating from zero to infinity,

$$
(9)\quad
\begin{aligned}
\frac{x^k \tanh x}{\sinh 2x \pm 2x} &= \frac{x^k}{\sinh 2x \pm 2x} \pm \frac{x^{k-1}(1 - e^{-2x})}{2(\sinh 2x \pm 2x)} \mp \frac{x^{k-1}e^{-x}}{2\cosh x},\\
\frac{x^k \coth x}{\sinh 2x \pm 2x} &= \frac{x^k}{\sinh 2x \pm 2x} \mp \frac{x^{k-1}(1 + e^{-2x})}{2(\sinh 2x \pm 2x)} \pm \frac{x^{k-1}e^{-x}}{2\sinh x}.
\end{aligned}
$$

The integrals become

$$
(10)\quad
\begin{aligned}
III_k &= I_k + (I_{k-1} - II_{k-1} - s_k)/k,\\
III_k^* &= I_k^* - (I^*_{k-1} - II^*_{k-1} - s_k)/k,\\
IV_k &= I_k - (I_{k-1} + II_{k-1} - S_k)/k,\\
IV_k^* &= I_k^* + (I^*_{k-1} + II^*_{k-1} - S_k)/k,
\end{aligned}
$$

where

$$
\begin{aligned}
s_k &= \frac{2^{k-1}}{(k-1)!}\int_0^\infty \frac{x^{k-1}e^{-x}dx}{\cosh x} = 1 - \frac{1}{2^k} + \frac{1}{3^k} - \frac{1}{4^k} + \cdots, \\
S_k &= \frac{2^{k-1}}{(k-1)!}\int_0^\infty \frac{x^{k-1}e^{-x}dx}{\sinh x} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots.
\end{aligned}
\tag{11}
$$

The values of s_k and S_k were tabulated by Glaisher [8].

Next, consider the four integrals V_k, V_k^*, VI_k, VI_k^*. By expanding $\sinh x$ into power series of x, the integral V_k develops into the series

$$V_k = \sum_{n=0}^{\infty} \binom{2n+k+1}{k} \frac{I_{2n+k+1}}{2^{2n+k+1}}. \tag{12}$$

In order to improve the convergence, Kummer transformation (Knopp [9]) may be used. The series then becomes

$$V_k = 1 - \frac{1}{3^{k+1}} - \sum_{n=0}^{\infty} \binom{2n+k+1}{k} \frac{1 - I_{2n+k+1}}{2^{2n+k+1}}. \tag{13}$$

Similarly, the integral V_k^* develops into the series

$$V_k^* = 1 - \frac{1}{3^{k+1}} + \sum_{n=0}^{\infty} \binom{2n+k+1}{k} \frac{I^*_{2n+k+1} - 1}{2^{2n+k+1}}. \tag{14}$$

By expanding $\cosh x$ into power series of x, the following series are obtained in the same way,

$$
\begin{aligned}
VI_k &= 1 + \frac{1}{3^{k+1}} - \sum_{n=0}^{\infty} \binom{2n+k}{k} \frac{1 - I_{2n+k}}{2^{2n+k}}, \\
VI_k^* &= 1 + \frac{1}{3^{k+1}} + \sum_{n=0}^{\infty} \binom{2n+k}{k} \frac{I^*_{2n+k} - 1}{2^{2n+k}}.
\end{aligned}
\tag{15}
$$

The last four integrals may be evaluated by splitting the integrands into the following and then integrating from zero to infinity,

$$
\begin{aligned}
\frac{x^k \tanh x \sinh x}{\sinh 2x \pm 2x} &= \frac{x^k \cosh x}{\sinh 2x \pm 2x} \pm \frac{x^{k-1}\sinh x}{\sinh 2x \pm 2x} \mp \frac{x^{k-1}}{2\cosh x}, \\
\frac{x^k \coth x \cosh x}{\sinh 2x \pm 2x} &= \frac{x^k \sinh x}{\sinh 2x \pm 2x} \mp \frac{x^{k-1}\cosh x}{\sinh 2x \pm 2x} \pm \frac{x^{k-1}}{2\sinh x}.
\end{aligned}
\tag{16}
$$

The integrals become

$$
\begin{aligned}
VII_k &= VI_k + (V_{k-1} - u_k)/k, \\
VII_k^* &= VI_k^* - (V^*_{k-1} - u_k)/k, \\
VIII_k &= V_k - (VI_{k-1} - U_k)/k, \\
VIII_k^* &= V_k^* + (VI^*_{k-1} - U_k)/k,
\end{aligned}
\tag{17}
$$

where

$$(18)\qquad \begin{aligned} u_k &= \frac{1}{2(k-1)!}\int_0^\infty \frac{x^{k-1}dx}{\cosh x} = 1 - \frac{1}{3^k} + \frac{1}{5^k} - \frac{1}{7^k} + \cdots, \\ U_k &= \frac{1}{2(k-1)!}\int_0^\infty \frac{x^{k-1}dx}{\sinh x} = 1 + \frac{1}{3^k} + \frac{1}{5^k} + \frac{1}{7^k} + \cdots. \end{aligned}$$

The values of U_k and u_k were also tabulated by Glaisher [8, 10].

The preceding method of evaluation is satisfactory as far as it goes, except that it does not cover the initial integrals III_0, III_1, III_2^*, III_3^*, VII_0, VII_1^*, and VII_2^*.

The integrals III_0 and III_1 may be evaluated by expanding $\tanh x$ into the following series and then integrating from zero to infinity,

$$(19)\qquad \tanh x = \frac{1}{\sinh 2x}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{(2n)!}.$$

This leads to

$$(20)\qquad \begin{aligned} III_0 &= \sum_{n=1}^{\infty}(U_{2n} - I_{2n-1})/2n, \\ III_1 &= \sum_{n=1}^{\infty}(U_{2n+1} - I_{2n}). \end{aligned}$$

An alternative expression for III_1 may be obtained as follows. By using the expansion

$$(21)\qquad \sinh x = \tanh x \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!},$$

the integral V_k develops into the series

$$(22)\qquad V_k = 1 + \frac{1}{3^{k+1}} - \sum_{n=0}^{\infty}\binom{2n+k}{k}\frac{1 - III_{2n+k}}{2^{2n+k}},$$

which gives when $k = 1$,

$$(23)\qquad III_1 = \frac{7}{9} - 2(1 - V_1) + \sum_{n=1}^{\infty}\frac{2n+1}{2^{2n}}(1 - III_{2n+1}).$$

The integrals III_2^* and III_3^* may be evaluated from the following series which is obtained in the same way,

$$(24)\qquad V_k^* = 1 + \frac{1}{3^{k+1}} + \sum_{n=0}^{\infty}\binom{2n+k}{k}\frac{III^*_{2n+k} - 1}{2^{2n+k}}.$$

When $k = 2$ and 3 respectively, this series gives

$$(25)\qquad \begin{aligned} III_2^* &= \frac{23}{27} + 4(V_2^* - 1) - \sum_{n=1}^{\infty}\binom{2n+2}{2}\frac{III^*_{2n+2} - 1}{2^{2n}}, \\ III_3^* &= \frac{73}{81} + 8(V_3^* - 1) - \sum_{n=1}^{\infty}\binom{2n+3}{3}\frac{III^*_{2n+3} - 1}{2^{2n}}. \end{aligned}$$

By expanding $\sinh x$ into power series of x, the integral VII_0 develops into the series

$$(26)\qquad VII_0 = \frac{2}{3} - \sum_{n=0}^{\infty}\frac{1 - III_{2n+1}}{2^{2n+1}}.$$

Similarly, the integral VII_k^* develops into the series

$$VII_k^* = 1 - \frac{1}{3^{k+1}} + \sum_{n=0}^{\infty} \binom{2n+k+1}{k} \frac{III^*_{2n+k+1} - 1}{2^{2n+k+1}}. \tag{27}$$

When $k = 1$ and 2 respectively, it gives

$$\begin{aligned} VII_1^* &= \frac{8}{9} + \sum_{n=1}^{\infty} \frac{2n}{2^{2n}} (III^*_{2n} - 1), \\ VII_2^* &= \frac{26}{27} + \sum_{n=1}^{\infty} \frac{2n(2n+1)}{2^{2n+2}} (III^*_{2n+1} - 1). \end{aligned} \tag{28}$$

Values of the twelve integrals thus computed and rounded off to six decimals, iusng Nelson's tables [7] when necessary, are given in Tables 1 and 2. The followngi formulas are useful for checking purposes:

$$\begin{aligned}
2(1 - III_1) + \sum_{k=1}^{\infty} (III^*_{2k+1} - III_{2k+1}) &= 2, \\
(1 - III_2) + \sum_{k=2}^{\infty} \frac{k}{2} (III^*_{2k} - III_{2k}) &= 1, \\
(1 - IV_2) + \sum_{k=2}^{\infty} \frac{k}{2} (IV^*_{2k} - IV_{2k}) &= 1, \\
(1 - IV_3) + \sum_{k=2}^{\infty} \frac{k(2k+1)}{6} (IV^*_{2k+1} - IV_{2k+1}) &= 1, \\
\sum_{k=1}^{\infty} k(VI_k - V_k) &= \tfrac{1}{2} I_1, \\
\sum_{k=0}^{\infty} (2k+1)(VI_{2k+1} - VII_{2k+1}) &= \tfrac{1}{2} I_1, \\
\sum_{k=1}^{\infty} k(VIII_{2k} - V_{2k}) &= \tfrac{1}{4} I_1, \\
\sum_{k=1}^{\infty} k(2k-1)(VI_{2k} - VII_{2k}) &= \tfrac{1}{4} I_2, \\
\sum_{k=1}^{\infty} \frac{k(2k+1)}{3} (VIII_{2k+1} - V_{2k+1}) &= \tfrac{1}{12} I_2, \\
\sum_{k=3}^{\infty} \frac{k(k-1)(k-2)}{3} (VI_k^* - V_k^*) &= \tfrac{1}{4} I_2^*, \\
\sum_{k=2}^{\infty} \frac{k(k-1)}{2} (V_k^* - VII_k^*) &= \tfrac{1}{4} III_2^*, \\
\sum_{k=4}^{\infty} \frac{k(k-1)(k-2)(k-3)}{24} (VIII_k^* - VI_k^*) &= \tfrac{1}{16} IV_4^*.
\end{aligned} \tag{29}$$

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Tables for the Rapid and Accurate Numerical Evaluation of Certain Infinite Integrals Involving Bessel Functions

Introduction. In a recent paper [1] the author has formulated a method based on Euler's transformation of slowly convergent alternating series for the numerical evaluation of integrals of the form $\int_a^\infty f(x)dx$, where a is a constant and where $f(x)$ oscillates about zero in such a way that the integral over each half-cycle is smaller in absolute magnitude than (and opposite in sign to) that over the preceding half-cycle. The author has had occasion to make much use of this method for the evaluation of integrals of the type

$$\int_0^\infty J_0(x)g(x)dx, \tag{1}$$

and

$$\int_0^\infty J_1(x)h(x)dx, \tag{2}$$

where $g(x)$, $h(x)$ are well-behaved continuous functions which tend to a finite constant value or zero as x tends to infinity; here $J_0(x)$, $J_1(x)$ are Bessel Functions (of the first kind) of orders zero and one, respectively. The present paper gives tables useful in the evaluation of (1), (2).

The author is grateful to Yigal Accad who performed most of the numerical calculations for Tables 1 and 2.

Description of the Method. The method employed [1] in evaluating integrals of the type (1) involved performing the integration over each of the first twenty half-cycles, i.e., evaluating the integrals

$$\int_{x_{i-1}}^{x_i} J_0(x)g(x)dx \quad i = 1, 2, \cdots, 20 \tag{3}$$

where x_0 is zero and x_i is the ith zero of $J_0(x)$. The first twenty terms of a slowly convergent alternating series for (1) were thus obtained. Euler's transformation (Bromwich [2], p. 62) was then applied to this series in order to obtain a rapidly convergent series for the numerical value of (1). Integrals of the type (2) were treated similarly.

In order to obtain high accuracy in the separate integrations (3) the Gauss quadrature formula (NBS AMS 37 [3]) was used for sixteen points of subdivision of each interval. Tables of abscissae and coefficients are given in [3] for $n = 2(1)16$, 15D, and in Davis and Rabinowitz [7] for $n = 2, 4, 8, 16, 20, 24, 32, 40, 48$, 20D. Values are also available in [7] for $n = 64$, 80 and 96. For a given value of n, Gauss' formula provides an approximation equivalent to replacing the integrand by a polynomial of degree $2n - 1$.

Description and Use of the Tables. Table 1 gives values of x and the corresponding values of $J_0(x)$. The values of x were obtained by computation from values given in [3] for the interval $(-1, 1)$ by means of the formula

$$x_i = x_i^1 \frac{q-p}{2} + \frac{q+p}{2}$$

where p, q are the lower and upper limits of integration, and in our case are zeros of $J_0(x)$ (the first value of p is zero), and the x_i^1 (given in [3]) refer to the interval $(-1, 1)$. The zeros of $J_0(x)$ are also given.

The values of $J_0(x)$ were obtained by interpolation from the Harvard University tables [4]. In view of the high accuracy needed interpolation was effected by means of the first four terms of Taylor's theorem

$$\begin{aligned} J_0(x+h) &\doteqdot J_0(x) + hJ_0'(x) + \frac{h^2}{2!}J_0''(x) + \frac{h^3}{3!}J_0'''(x) \\ &= J_0(x) - hJ_1(x) + \tfrac{1}{2}h^2\left(\frac{J_1(x)}{x} - J_0(x)\right) \\ &\qquad + \tfrac{1}{6}h^3\left(J_1(x) + \frac{J_0(x)}{x} - \frac{2J_1(x)}{x^2}\right), \end{aligned}$$

these terms being sufficient for 10 place accuracy with the maximum value of h required. The tables are believed to be accurate to within a few units in the tenth place of decimals.

Table 2 was similarly prepared for $J_1(x)$. Here the interpolation for $J_1(x)$

was effected from [4] by means of the Taylor's theorem expansion

$$J_1(x+h) \doteqdot J_1(x) + h\left(J_0(x) - \frac{J_1(x)}{x}\right) + \tfrac{1}{2}h^2\left(\frac{2J_1(x)}{x^2} - \frac{J_0(x)}{x} - J_1(x)\right) + \tfrac{1}{6}h^3\left(\frac{3J_0(x)}{x^2} - \frac{6J_1(x)}{x^3} + \frac{2J_1(x)}{x} - J_0(x)\right).$$

In Tables 1 and 2 the zeros of $J_0(x)$, $J_1(x)$ were taken from tables in BAASMTC [5].

Table 3 gives the appropriate integration coefficients (these are quoted from

TABLE 3. *Gauss Integration Coefficients*

.02715 24594
.06225 35239
.09515 85117
.12462 89713
.14959 59888
.16915 65194
.18260 34150
.18945 06105
.18945 06105
.18260 34150
.16915 65194
.14959 59888
.12462 89713
.09515 85117
.06225 35239
.02715 24594

[3] for the interval $(-1, 1)$). After using them the result must be multiplied by half the length of the interval in each case.

Method of Checking. The tables were checked as follows:

Table 1 was checked by using it to calculate

$$\int_{x_{i-1}}^{x_i} xJ_0(x)dx$$

over each half-cycle and comparing the result with the known value of the integral

$$[xJ_1(x)]_{x_{i-1}}^{x_i} = x_iJ_1(x_i) - x_{i-1}J_1(x_{i-1}),$$

the values of $J_1(x_i)$ being obtained from [5]. In each case agreement was obtained to within the accuracy warranted by the number of significant figures in the table.

Table 2 was checked by using it to calculate

$$\int_{y_{i-1}}^{y_i} J_1(x)dx = [-J_0(x)]_{y_{i-1}}^{y_i} = J_0(y_{i-1}) - J_0(y_i)$$

over each half-cycle, where the y_i are the zeros of $J_1(x)$. The results were calculated to ten places of decimals and in each case were accurate to within two units in the last place.

TABLE 1

	x	$J_0(x)$		x	$J_0(x)$
	.00000 00000			5.52007 81103	
1	.01274 44512	.99995 93952	1	5.53668 49893	.00564 19942
2	.06664 37005	.99888 99624	2	5.60691 93443	.02928 21569
3	.16156 67593	.99348 46849	3	5.73061 04883	.06978 62151
4	.29410 48650	.97849 22054	4	5.90331 65740	.12301 12685
5	.45947 04869	.94791 40344	5	6.11879 91260	.18206 24818
6	.65168 75524	.89661 10444	6	6.36927 09191	.23760 14372
7	.86380 90708	.82197 96824	7	6.64567 95559	.27944 55482
8	1.08816 85230	.72517 29892	8	6.93803 50453	.29921 03673
9	1.31665 70348	.61136 06080	9	7.23577 09779	.29294 41222
10	1.54101 64869	.48883 00670	10	7.52812 64673	.26243 80011
11	1.75313 80053	.36721 18144	11	7.80453 51041	.21449 26872
12	1.94535 50709	.25549 12057	12	8.05500 68972	.15853 58154
13	2.11072 06927	.16052 05519	13	8.27048 94492	.10381 91971
14	2.24325 87984	.08645 02192	14	8.44319 55349	.05742 97048
15	2.33818 18573	.03506 03860	15	8.56688 66789	.02366 25625
16	2.39208 11065	.00663 36641	16	8.63712 10339	.00451 20991
	2.40482 55577			8.65372 79129	
1	2.42133 49398	−.00854 11306	1	8.67035 68206	−.00450 94160
2	2.49115 69581	−.04397 83230	2	8.74068 43400	−.02345 71287
3	2.61412 19276	−.10342 59686	3	8.86453 95579	−.05612 15388
4	2.78581 40782	−.17919 07027	4	9.03747 47347	−.09944 20369
5	3.00003 15602	−.26006 26555	5	9.25324 31200	−.14809 16553
6	3.24903 28661	−.33251 74168	6	9.50404 71589	−.19457 96817
7	3.52381 87438	−.38328 09269	7	9.78082 24462	−.23045 12070
8	3.81445 78522	−.40269 93971	8	10.07356 57393	−.24845 57197
9	4.11044 58158	−.38757 11113	9	10.37169 66127	−.24484 61857
10	4.40108 49242	−.34203 66196	10	10.66443 99058	−.22066 46191
11	4.67587 08019	−.27600 29383	11	10.94121 51931	−.18130 82267
12	4.92487 21078	−.20186 70491	12	11.19201 92320	−.13461 51679
13	5.13908 95898	−.13110 13628	13	11.40778 76173	−.08848 00142
14	5.31078 17404	−.07207 25099	14	11.58072 27941	−.04908 31439
15	5.43374 67099	−.02957 09059	15	11.70457 80120	−.02026 32192
16	5.50356 87282	−.00562 57234	16	11.77490 55314	−.00386 81029
	5.52007 81103			11.79153 44391	

TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	11.79153 44391			18.07106 39679	
1	11.80817 17029	.00386 45915	1	18.08770 75348	.00312 28948
2	11.87853 45618	.02012 43811	2	18.15809 70515	.01627 88282
3	12.00245 20168	.04823 70070	3	18.28206 14541	.03908 92889
4	12.17547 40932	.08568 63326	4	18.45514 90820	.06960 69216
5	12.39135 09018	.12799 26729	5	18.67110 76781	.10428 43725
6	12.64228 09694	.16873 86824	6	18.92213 28135	.13794 52734
7	12.91919 53359	.20055 59550	7	19.19915 20923	.16454 42527
8	13.21208 57321	.21699 70655	8	19.49215 34535	.17868 44763
9	13.51036 64157	.21458 33409	9	19.79054 71443	.17733 10142
10	13.80325 68119	.19401 42035	10	20.08354 85055	.16087 71698
11	14.08017 11784	.15987 41093	11	20.36056 77843	.13297 88950
12	14.33110 12460	.11900 00109	12	20.61159 29197	.09924 99767
13	14.54697 80546	.07837 94720	13	20.82755 15158	.06551 96366
14	14.72000 01310	.04355 03186	14	21.00063 91437	.03646 96948
15	14.84391 75860	.01799 93958	15	21.12460 35463	.01509 18172
16	14.91428 04449	.00343 81248	16	21.19499 30630	.00288 47612
	14.93091 77086			21.21163 66299	
1	14.94755 90159	−.00343 51301	1	21.22828 15866	−.00288 27393
2	15.01793 89756	−.01789 91458	2	21.29867 69809	−.01503 12702
3	15.14188 65473	−.04294 96191	3	21.42265 17345	−.03611 16663
4	15.31495 06747	−.07640 67698	4	21.59575 38153	−.06434 88934
5	15.53087 99495	−.11433 61880	5	21.81173 04440	−.09648 81035
6	15.78187 10027	−.15103 88400	6	22.06277 65401	−.12775 47593
7	16.05885 26700	−.17990 39278	7	22.33981 89501	−.15254 57750
8	16.35181 42497	−.19507 68842	8	22.63284 47770	−.16583 05446
9	16.65016 74269	−.19331 88429	9	22.93126 33838	−.16474 72215
10	16.94312 90066	−.17514 01558	10	23.22428 92107	−.14961 08067
11	17.22011 06739	−.14458 58966	11	23.50133 16207	−.12378 03876
12	17.47110 17271	−.10779 31239	12	23.75237 77168	−.09245 99613
13	17.68703 10019	−.07109 30831	13	23.96835 43455	−.06107 91462
14	17.86009 51293	−.03954 30642	14	24.14145 64263	−.03401 63997
15	17.98404 27010	−.01635 51985	15	24.26543 11799	−.01408 19757
16	18.05442 26607	−.00312 53477	16	24.33582 65742	−.00269 23112
	18.07106 39679			24.35247 15308	

TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	24.35247 15308			30.63460 64684	
1	24.36911 74027	.00269 06195	1	30.65125 34327	.00239 91633
2	24.43951 66677	.01403 25020	2	30.72165 73180	.01251 61523
3	24.56349 82381	.03372 47328	3	30.84564 70254	.03009 59796
4	24.73660 98370	.06012 63461	4	31.01876 99856	.05369 50891
5	24.95259 83413	.09021 33604	5	31.23477 26651	.08063 46686
6	25.20365 82413	.11953 24224	6	31.48584 90422	.10694 79499
7	25.48071 58846	.14283 84354	7	31.76292 48688	.12793 92673
8	25.77375 78237	.15540 20499	8	32.05598 60401	.13934 93292
9	26.07219 28391	.15450 96015	9	32.35444 06419	.13870 48969
10	26.36523 47782	.14042 06959	10	32.64750 18132	.12619 35228
11	26.64229 24215	.11625 87219	11	32.92457 76398	.10458 43845
12	26.89335 23215	.08689 57273	12	33.17565 40169	.07823 96934
13	27.10934 08258	.05743 36567	13	33.39165 66964	.05175 15312
14	27.28245 24247	.03199 94405	14	33.56477 96566	.02885 08462
15	27.40643 39951	.01325 09005	15	33.68876 93640	.01195 21353
16	27.47683 32601	.00253 38391	16	33.75917 32493	.00228 60352
	27.49347 91320			33.77582 02136	
1	27.51012 56384	−.00253 24019	1	33.79246 75193	−.00228 49556
2	27.58052 75870	−.01320 95154	2	33.86287 28479	−.01192 16140
3	27.70451 38837	−.03175 59526	3	33.98686 50970	−.02867 17157
4	27.87763 20816	−.05663 87732	4	34.15999 16062	−.05116 72201
5	28.09362 88193	−.08502 21095	5	34.37599 87138	−.07686 29839
6	28.34469 82897	−.11271 68644	6	34.62708 02379	−.10198 26451
7	28.62176 64945	−.13477 53040	7	34.90416 17446	−.12204 77212
8	28.91481 96043	−.14672 13774	8	35.19722 89237	−.13298 71140
9	29.21326 59961	−.14596 96665	9	35.49568 96437	−.13242 66679
10	29.50631 91059	−.13273 89002	10	35.78875 68228	−.12052 95168
11	29.78338 73107	−.10995 96619	11	36.06583 83295	−.09992 72078
12	30.03445 67811	−.08222 81150	12	36.31691 98536	−.07478 03172
13	30.25045 35188	−.05437 12708	13	36.53292 69612	−.04947 71799
14	30.42357 17167	−.03030 32039	14	36.70605 34704	−.02758 90427
15	30.54755 80134	−.01255 14239	15	36.83004 57195	−.01143 12041
16	30.61795 99620	−.00240 04013	16	36.90045 10481	−.00218 65932
	30.63460 64684			36.91709 83537	

TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	36.91709 83537			43.19979 17132	
1	36.93374 59205	.00218 56417	1	43.21643 96470	.00202 05537
2	37.00415 23534	.01140 44585	2	43.28684 76319	.01054 45035
3	37.12814 65473	.02743 21950	3	43.41084 45592	.02536 98152
4	37.30127 57720	.04896 56861	4	43.58397 76004	.04529 96158
5	37.51728 62677	.07357 53320	5	43.79999 28578	.06809 50035
6	37.76837 17300	.09765 02107	6	44.05108 38550	.09041 97790
7	38.04545 75826	.11690 15020	7	44.32817 58158	.10830 19105
8	38.33852 93584	.12742 34635	8	44.62125 40521	.11811 38815
9	38.63699 47599	.12693 02235	9	44.91972 60329	.11772 07481
10	38.93006 65358	.11556 53707	10	45.21280 42692	.10723 69760
11	39.20715 23884	.09584 13189	11	45.48989 62300	.08897 80881
12	39.45823 78507	.07174 25318	12	45.74098 72272	.06663 43209
13	39.67424 83464	.04747 84574	13	45.95700 24846	.04411 43990
14	39.84737 75711	.02647 94775	14	46.13013 55258	.02461 06003
15	39.97137 17650	.01097 29247	15	46.25413 24531	.01020 06291
16	40.04177 81979	.00209 90896	16	46.32454 04380	.00195 15847
	40.05842 57646			46.34118 83717	
1	40.07507 35355	−.00209 82432	1	46.35783 64373	−.00195 08992
2	40.14548 08322	−.01094 92382	2	46.42824 49795	−.01018 15645
3	40.26947 65473	−.02634 06534	3	46.55224 28884	−.02449 89606
4	40.44260 78959	−.04702 58304	4	46.72537 73001	−.04375 05072
5	40.65862 10416	−.07067 63342	5	46.94139 42674	−.06577 72703
6	40.90970 95842	−.09382 67282	6	47.19248 72523	−.08735 88503
7	41.18679 88362	−.11235 56730	7	47.46958 14065	−.10465 74117
8	41.47987 42074	−.12250 42137	8	47.76266 19629	−.11416 40209
9	41.77834 32704	−.12206 57451	9	48.06113 63063	−.11380 89156
10	42.07141 86416	−.11116 79231	10	48.35421 68627	−.10369 54948
11	42.34850 78936	−.09221 87488	11	48.63131 10169	−.08605 66385
12	42.59959 64362	−.06904 71411	12	48.88240 40018	−.06445 79246
13	42.81560 95819	−.04570 38541	13	49.09842 09691	−.04267 99914
14	42.98874 09305	−.02549 38175	14	49.27155 53808	−.02381 32307
15	43.11273 66456	−.01056 56711	15	49.39555 32897	−.00987 09780
16	43.18314 39423	−.00202 13118	16	49.46596 18319	−.00188 86072
	43.19979 17132			49.48260 98974	

TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	49.48260 98974			55.76551 07550	
1	49.49925 80712	.00188 79842	1	55.78215 90943	.00177 84922
2	49.56966 70711	.00985 36907	2	55.85256 87941	.00928 29778
3	49.69366 57858	.02371 20292	3	55.97656 87414	.02234 17954
4	49.86680 13228	.04235 01575	4	56.14970 59995	.03991 06704
5	50.08281 96942	.06368 11286	5	56.36572 65182	.06002 74072
6	50.33391 43111	.08458 90824	6	56.61682 36311	.07975 79441
7	50.61101 02663	.10135 76521	7	56.89392 23408	.09559 78634
8	50.90409 27276	.11058 55944	8	57.18700 77154	.10433 45986
9	51.20256 90110	.11026 27587	9	57.48548 69658	.10406 33501
10	51.49565 14723	.10048 31520	10	57.77857 23404	.09486 31094
11	51.77274 74275	.08340 52240	11	58.05567 10501	.07876 32823
12	52.02384 20444	.06248 17090	12	58.30676 81630	.05901 96906
13	52.23986 04158	.04137 69630	13	58.52278 86817	.03909 30391
14	52.41299 59528	.02308 86479	14	58.69592 59398	.02181 80708
15	52.53699 46675	.00957 13465	15	58.81992 58871	.00904 57744
16	52.60740 36674	.00183 13567	16	58.89033 55869	.00173 09189
	52.62405 18411			58.90698 39261	
1	52.64070 01048	−.00183 07874	1	58.92363 23295	−.00173 04364
2	52.71110 94851	−.00955 55769	2	58.99404 23003	−.00903 24503
3	52.83510 88697	−.02299 63548	3	59.11804 27250	−.02174 01207
4	53.00824 53420	−.04107 61891	4	59.29118 06495	−.03883 90402
5	53.22426 48803	−.06177 33832	5	59.50720 19998	−.05842 15607
6	53.47536 08536	−.08206 70662	6	59.75830 00793	−.07763 33587
7	53.75245 83057	−.09835 15046	7	60.03539 98556	−.09306 32526
8	54.04554 23502	−.10732 38205	8	60.32848 63585	−.10158 19737
9	54.34402 02460	−.10702 86465	9	60.62696 67579	−.10133 15847
10	54.63710 42905	−.09755 19386	10	60.92005 32608	−.09238 50042
11	54.91420 17426	−.08098 46590	11	61.19715 30371	−.07671 51960
12	55.16529 77159	−.06067 67598	12	61.44825 11166	−.05749 13542
13	55.38131 72542	−.04018 64126	13	61.66427 24669	−.03808 43044
14	55.55445 37265	−.02242 64133	14	61.83741 03914	−.02125 66871
15	55.67845 31111	−.00929 74390	15	61.96141 08161	−.00881 34966
16	55.74886 24914	−.00177 90152	16	62.03182 07869	−.00168 65234
	55.76551 07550			62.04846 91902	

Table 2

	x	$J_1(x)$		x	$J_1(x)$
	.00000 00000			7.01558 66698	
1	.02030 62503	.01015 26018	1	7.03232 19654	.00501 63154
2	.10618 61074	.05301 82575	2	7.10309 94235	.02606 88611
3	.25743 08620	.12765 21143	3	7.22774 70376	.06226 85746
4	.46860 91943	.22793 16731	4	7.40178 86125	.11009 66693
5	.73209 29378	.34206 47543	5	7.61893 74325	.16354 80168
6	1.03836 01743	.45228 19201	6	7.87134 60560	.21430 84782
7	1.37634 19817	.53758 52259	7	8.14989 20857	.25311 94883
8	1.73382 29846	.57949 53889	8	8.44450 82807	.27216 07618
9	2.09788 29856	.56851 14490	9	8.74454 65243	.26752 21198
10	2.45536 39885	.50780 82279	10	9.03916 27193	.24053 22625
11	2.79334 57959	.41190 88809	11	9.31770 87490	.19721 25700
12	3.09961 30324	.30107 17052	12	9.57011 73725	.14615 17084
13	3.36309 67759	.19458 39510	13	9.78726 61925	.09591 29504
14	3.57427 51082	.10623 42640	14	9.96130 77674	.05314 08898
15	3.72551 98628	.04329 54722	15	10.08595 53815	.02191 91616
16	3.81139 97199	.00819 97562	16	10.15673 28396	.00418 21325
	3.83170 59702			10.17346 81351	
1	3.84857 90494	−.00678 05723	1	10.19016 28485	−.00416 51430
2	3.91993 92273	−.03509 12559	2	10.26076 86758	−.02167 90724
3	4.04561 30809	−.08323 73938	3	10.38511 40273	−.05192 08365
4	4.22108 75619	−.14584 38197	4	10.55873 35623	−.09212 85148
5	4.44002 41954	−.21441 93722	5	10.77535 58100	−.13743 55137
6	4.69451 09307	−.27792 36119	6	11.02715 23585	−.18092 68681
7	4.97535 02648	−.32471 41295	7	11.30502 29316	−.21471 99134
8	5.27239 20719	−.34554 21188	8	11.59892 47008	−.23197 38712
9	5.57490 05681	−.33641 08609	9	11.89823 53706	−.22906 10599
10	5.87194 23752	−.29986 78712	10	12.19213 71398	−.20682 24276
11	6.15278 17093	−.24400 22677	11	12.47000 77129	−.17021 73009
12	6.40726 84446	−.17965 91930	12	12.72180 42614	−.12656 04959
13	6.62620 50781	−.11727 67942	13	12.93842 65091	−.08328 24257
14	6.80167 95591	−.06471 02564	14	13.11204 60441	−.04624 08298
15	6.92735 34127	−.02661 42650	15	13.23639 13956	−.01910 14741
16	6.99871 35906	−.00506 97404	16	13.30699 72229	−.00364 75651
	7.01558 66698			13.32369 19363	

TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	13.32369 19363			19.61585 85105	
1	13.34036 92371	.00363 92005	1	19.63252 14378	.00299 89710
2	13.41090 14225	.01895 70437	2	19.70299 28337	.01563 52051
3	13.53511 70816	.04546 56026	3	19.82710 14354	.03755 37294
4	13.70855 55312	.08082 85751	4	20.00039 04040	.06689 70210
5	13.92495 18418	.12085 59397	5	20.21660 02095	.10026 96775
6	14.17648 57662	.15950 88072	6	20.46791 73449	.13270 31954
7	14.45406 65200	.18981 34949	7	20.74525 88608	.15837 97464
8	14.74766 17494	.20562 58827	8	21.03860 10500	.17208 93345
9	15.04666 02378	.20358 27840	9	21.33734 18412	.17088 33480
10	15.34025 54672	.18427 60601	10	21.63068 40304	.15511 21291
11	15.61783 62210	.15200 47172	11	21.90802 55463	.12827 75447
12	15.86937 01454	.11324 27689	12	22.15934 26817	.09578 30326
13	16.08576 64560	.07464 19546	13	22.37555 24872	.06325 40801
14	16.25920 49056	.04149 70116	14	22.54884 14558	.03521 86914
15	16.38342 05647	.01715 74777	15	22.67295 00575	.01457 70470
16	16.45395 27501	.00327 80206	16	22.74342 14534	.00278 66763
	16.47063 00509			22.76008 43806	
1	16.48729 82913	−.00327 29251	1	22.77674 39258	−.00278 40720
2	16.55779 21583	−.01705 76459	2	22.84720 10186	−.01451 84126
3	16.68194 03340	−.04094 59048	3	22.97128 44308	−.03488 63684
4	16.85528 45588	−.07288 03022	4	23.14453 82281	−.06218 23121
5	17.07156 33065	−.10912 92847	5	23.36070 41509	−.09327 05856
6	17.32296 05786	−.14426 61835	6	23.61197 02781	−.12354 19341
7	17.60039 05295	−.17197 26270	7	23.88925 55038	−.14757 67318
8	17.89382 62558	−.18662 79783	8	24.18253 81552	−.16049 77977
9	18.19266 23056	−.18509 40367	9	24.48121 83130	−.15951 75470
10	18.48609 80319	−.16781 59539	10	24.77450 09644	−.14492 09337
11	18.76352 79828	−.13863 52301	11	25.05178 61901	−.11994 54048
12	19.01492 52549	−.10341 91552	12	25.30305 23173	−.08962 51276
13	19.23120 40026	−.06824 25400	13	25.51921 82401	−.05922 29479
14	19.40454 82274	−.03797 23479	14	25.69247 20374	−.03298 98365
15	19.52869 64031	−.01570 98187	15	25.81655 54496	−.01365 90977
16	19.59919 02701	−.00300 24768	16	25.88701 25424	−.00261 16864
	19.61585 85105			25.90367 20876	

TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	25.90367 20876			32.18967 99110	
1	25.92032 93474	.00260 96493	1	32.20633 43684	.00234 09996
2	25.99077 67744	.01361 13779	2	32.27676 99432	.01221 33904
3	26.11484 31639	.03271 75400	3	32.40081 54597	.02937 08010
4	26.28807 31930	.05834 28100	4	32.57401 63444	.05240 83042
5	26.50420 94607	.08755 98843	5	32.79011 62490	.07871 53604
6	26.75544 11174	.11605 08696	6	33.04130 56382	.10442 22839
7	27.03268 83032	.13872 26066	7	33.31850 61796	.12494 39101
8	27.32593 07199	.15097 45446	8	33.61169 92609	.13611 63937
9	27.62456 99027	.15015 74510	9	33.91028 82003	.13551 63765
10	27.91781 23194	.13650 89575	10	34.20348 12816	.12331 84545
11	28.19505 95052	.11305 33742	11	34.48068 18230	.10222 15009
12	28.44629 11619	.08452 19696	12	34.73187 12122	.07648 52251
13	28.66242 74296	.05587 69690	13	34.94797 11168	.05059 84188
14	28.83565 74587	.03113 74830	14	35.12117 20015	.02821 12428
15	28.95972 38482	.01289 55290	15	35.24521 75180	.01168 81140
16	29.03017 12752	.00246 60528	16	35.31565 30928	.00223 56390
	29.04682 85349			35.33230 75501	
1	29.06348 41780	−.00246 43999	1	35.34896 11121	−.00223 44653
2	29.13392 47675	−.01285 56731	2	35.41939 29002	−.01165 87150
3	29.25797 91152	−.03090 89818	3	35.54343 17477	−.02804 16792
4	29.43119 23308	−.05513 72476	4	35.71662 33207	−.05004 84236
5	29.64730 76205	−.08278 50383	5	35.93271 16074	−.07519 26901
6	29.89851 48930	−.10977 68854	6	36.18388 74921	−.09978 23087
7	30.17573 51694	−.13129 35471	7	36.46107 31306	−.11943 51446
8	30.46894 91244	−.14296 90393	8	36.75425 04492	−.13016 38763
9	30.76755 93216	−.14227 43786	9	37.05282 33358	−.12963 88064
10	31.06077 32766	−.12941 16541	10	37.34600 06544	−.11801 27490
11	31.33799 35530	−.10722 87815	11	37.62318 62929	−.09785 65260
12	31.58920 08255	−.08020 27911	12	37.87436 21776	−.07324 13135
13	31.80531 61152	−.05304 14604	13	38.09045 04643	−.04846 48482
14	31.97852 93308	−.02956 61656	14	38.26364 20373	−.02702 71631
15	32.10258 36785	−.01224 73498	15	38.38768 08848	−.01119 91603
16	32.17302 42680	−.00234 23778	16	38.45811 26729	−.00214 22896
	32.18967 99110			38.47476 62348	

TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	38.47476 62348			44.75931 89977	
1	38.49141 91040	.00214 12735	1	44.77597 08809	.00198 52730
2	38.56184 79626	.01117 33773	2	44.84639 55686	.01036 06792
3	38.68588 16508	.02687 81611	3	44.97042 19116	.02492 87731
4	38.85906 60202	.04798 12388	4	45.14359 60250	.04451 51717
5	39.07514 53189	.07210 44757	5	45.35966 25275	.06692 15430
6	39.32631 07563	.09571 08701	6	45.61081 30909	.08887 03803
7	39.60348 48656	.11459 65682	7	45.88797 07859	.10645 75916
8	39.89664 99899	.12493 01313	8	46.18111 85490	.11611 55971
9	40.19521 04577	.12446 56086	9	46.47966 13360	.11574 22770
10	40.48837 55820	.11333 82116	10	46.77280 90991	.10544 63168
11	40.76554 96913	.09400 72264	11	47.04996 67941	.08750 13044
12	41.01671 51287	.07037 82521	12	47.30111 73575	.06553 43925
13	41.23279 44274	.04658 04345	13	47.51718 38600	.04338 95839
14	41.40597 87968	.02598 07709	14	47.69035 79734	.02420 77320
15	41.53001 24850	.01076 68903	15	47.81438 43164	.01003 40865
16	41.60044 13436	.00205 97434	16	47.88480 90041	.00191 97690
	41.61709 42128			47.90146 08872	
1	41.63374 65353	−.00205 88518	1	47.91811 24121	−.00191 90607
2	41.70417 30810	−.01074 40331	2	47.98853 55848	−.01001 56479
3	41.82820 26960	−.02584 84372	3	48.11255 92598	−.02410 07718
4	42.00138 13780	−.04615 08277	4	48.28572 96479	−.04304 20054
5	42.21745 35808	−.06936 81412	5	48.50179 15025	−.06471 68956
6	42.46861 07699	−.09210 05548	6	48.75293 66631	−.08595 79773
7	42.74577 57769	−.11030 23920	7	49.03008 83960	−.10298 88602
8	43.03893 12737	−.12028 11784	8	49.32322 98530	−.11235 50074
9	43.33748 19369	−.11986 63996	9	49.62176 62178	−.11201 66719
10	43.63063 74337	−.10917 88005	10	49.91490 76748	−.10207 23609
11	43.90780 24407	−.09057 93974	11	50.19205 94077	−.08471 72272
12	44.15895 96298	−.06782 68662	12	50.44320 45683	−.06345 97884
13	44.37503 18326	−.04490 01454	13	50.65926 64229	−.04202 19605
14	44.54821 05146	−.02504 72789	14	50.83243 68110	−.02344 73534
15	44.67224 01296	−.01038 11283	15	50.95646 04860	−.00971 96893
16	44.74266 66753	−.00198 60641	16	51.02688 36587	−.00185 97015
	44.75931 89977			51.04353 51836	

TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	51.04353 51836			57.32752 54379	
1	51.06018 64126	.00185 90619	1	57.34417 62108	.00175 42236
2	51.13060 83339	.00970 29509	2	57.41459 62035	.00915 64627
3	51.25462 98048	.02335 01746	3	57.53861 42778	.02203 79680
4	51.42779 71154	.04170 60842	4	57.71177 68459	.03936 95712
5	51.64385 51303	.06271 67691	5	57.92782 89436	.05921 66524
6	51.89499 58277	.08331 44272	6	58.17896 27631	.07868 54349
7	52.17214 26353	.09983 86147	7	58.45610 19805	.09431 85737
8	52.46527 88827	.10893 77477	8	58.74923 01998	.10294 55141
9	52.76380 99421	.10862 92546	9	59.04775 30834	.10268 50402
10	53.05694 61895	.09900 29593	10	59.34088 13027	.09361 30005
11	53.33409 29971	.08218 31360	11	59.61802 05201	.07773 02633
12	53.58523 36945	.06157 05817	12	59.86915 43396	.05824 89261
13	53.80129 17094	.04077 60577	13	60.08520 64373	.03858 43703
14	53.97445 90200	.02275 44296	14	60.25836 90054	.02153 50060
15	54.09848 04909	.00943 31183	15	60.38238 70797	.00892 86590
16	54.16890 24122	.00180 49432	16	60.45280 70724	.00170 85351
	54.18555 36411			60.46945 78453	
1	54.20220 46228	−.00180 43618	1	60.48610 84404	−.00170 80464
2	54.27262 54982	−.00941 78326	2	60.55652 76810	−.00891 57171
3	54.39664 51273	−.02266 56242	3	60.68054 44308	−.02145 97360
4	54.56980 98663	−.04048 73357	4	60.85370 51497	−.03833 95708
5	54.78586 46726	−.06089 13712	5	61.06975 49401	−.05767 29573
6	55.03700 16405	−.08090 07468	6	61.32088 60776	−.07664 27230
7	55.31414 43323	−.09696 09091	7	61.59802 23352	−.09188 11714
8	55.60727 62264	−.10581 45509	8	61.89114 74241	−.10029 79275
9	55.90580 28526	−.10553 17575	9	62.18966 71197	−.10005 69840
10	56.19893 47467	−.09619 48735	10	62.48279 22086	−.09122 85052
11	56.47607 74385	−.07986 37153	11	62.75992 84662	−.07575 91800
12	56.72721 44064	−.05984 06981	12	63.01105 96037	−.05677 78096
13	56.94326 92127	−.03963 48225	13	63.22710 93941	−.03761 32667
14	57.11643 39517	−.02211 95390	14	63.40027 01130	−.02099 45050
15	57.24045 35808	−.00917 04951	15	63.52428 68628	−.00870 50043
16	57.31087 44562	−.00175 47553	16	63.59470 61034	−.00166 57858
	57.32752 54379			63.61135 66985	

Example of the Use of the Tables. The example

$$\int_0^\infty J_0(x)dx = 1$$

given in [1] is repeated here partly since an error in Watson ([6], p. 752) is revealed, and partly as an extension of the tabulation of

$$\int_{x_n}^{x_{n+1}} J_0(x)dx$$

to a larger value of n than that obtainable from Watson ([6], p. 752).

Applying the integration coefficients directly to the values of $J_0(x)$ in Table 1 we get

n	$\pm\int_{x_{n-1}}^{x_n} J_0(x)dx$		Δ^2		Δ^4		Δ^6		Δ^8	
1	1.47030 0043									
2	.80145 4213									
3	.59932 2516									
4	.49904 9621									
5	.43653 5113									
6	.39282 2560									
7	.36005 6836									
8	.33432 0981									
9	.31341 5072									
10	.29599 6007									
11	.28119 1319									
		−12784 903								
12	.26840 6416		15987 49							
		−11186 154		−30760 6						
13	.25722 0262		12911 43		76952					
		− 9895 011		−23065 4		−23112				
14	.24732 5251		10604 89		53840		7965			
		− 8834 522		−17681 4		−15147		−3058		
15	.23849 0729		8836 75		38693		4907		1263	
		− 7950 847		−13812 1		−10240		−1795		−489
16	.23053 9882		7455 54		28453		3112		774	
		− 7205 293		−10966 8		− 7128		−1021		
17	.22333 4589		6358 86		21325		2091			
		− 6569 407		− 8834 3		− 5037				
18	.21676 5182		5475 43		16288					
		− 6021 864		− 7205 5						
19	.21074 3318		4754 88							
		− 5546 376								
20	.20519 6942									

Here x_0 denotes 0.

Now, choosing the first advancing row of differences we have (by Euler's transformation) approximately

$$\begin{aligned}\int_0^\infty J_0(x)dx = {} & 1.47030\ 0043 - 0.80145\ 4213 + .59932\ 2516 - .49904\ 9621 \\ & + .43653\ 5113 - .39282\ 2560 + .36005\ 6836 - .33432\ 0981 \\ & + .31341\ 5072 - .29599\ 6007 + (1/2)(0.28119\ 1319) \\ & + (1/4)(0.01278\ 4903) + (1/8)(0.00159\ 8749) \\ & + (1/16)(0.00030\ 7606) + (1/32)(0.00007\ 6952) \\ & + (1/64)(0.00002\ 3112) + (1/128)(0.00000\ 7965) \\ & + (1/256)(0.00000\ 3058) + (1/512)(0.00000\ 1263) \\ & + (1/1024)(0.00000\ 0489) = 0.99999\ 9992.\end{aligned}$$

It should be noted that the first figure in the table

$$1.47030\ 00434 = \int_0^{x_1} J_0(x)dx \tag{4}$$

disagrees with the figure for this integral quoted in [1], where it is taken from Watson ([6], page 752). This integral was therefore calculated independently by interpolation,

$$\int_0^{x+h} J_0(t)dt = \int_0^x J_0(t)dt + hJ_0(x) - \frac{h^2}{2!}J_1(x) + \frac{h^3}{3!}\left(\frac{J_1(x)}{x} - J_0(x)\right),$$

(Taylor's theorem) using tables of $J_0(x)$, $J_1(x)$ [4] and a table [3] of $\int_0^x J_0(t)dt$. For this x was taken to be 2.4 and h to be 0.00482 55577, so that $x + h = 2.40482\ 55577$ is the first zero of $J_0(x)$. The result obtained agreed with (4) to ten decimal places, and this confirmation reveals an error in Watson [6] where the value

$$\frac{1}{2}\int_0^{x_1} J_0(x)dx = 0.73522\ 08$$

is quoted. This value should be corrected to 0.73515 00.

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On the Time-Step to be Used for the Computation of Orbits by Numerical Integration

W. J. Eckert [1] has adapted Cowell's method of numerical integration to the determination of orbits on punched card machines. Eckert, Brouwer, and Clemence [2] have used Cowell's method on a large-scale computer. We shall assume that Cowell's method as modified by Eckert is the method of numerical integration best suited for the determination of orbits on large-scale computers. What we are concerned with in this paper is how to make use of this method in such a way that we do the least work (have the fewest arithmetic operations) per unit of advance in the time. Our results apply also to the computation of orbits on ordinary desk computers, but additional factors, such as the work involved in transcribing, must be considered.

In Cowell's method as modified by Eckert the attractions and the central differences of the attractions which are required at the "new time-step" are obtained from the values at previous time-steps by an extrapolation process. This extrapolation process may be shown to be equivalent to extrapolating the attractions ahead in the time using Newton's backward difference interpolation formula, so that Cowell's integration formula may be written in terms of backward differences instead of central differences. Let $x_n{}^i$, $i = 1, 2, 3$, be rectangular coordinates and let $X_n{}^i$ be the attractions at time $t = n\Delta t$. Then we obtain for $x^i{}_{n+1}$ in terms of backward differences,

$$
\begin{aligned}
(1)\qquad x^i{}_{n+1} = {}''X^i{}_{n+1} + \Delta t^2 \Bigg(& \frac{1}{12} X_n{}^i + \frac{1}{12} \nabla X_n{}^i + \frac{19}{240} \nabla^2 X_n{}^i + \frac{3}{40} \nabla^3 X_n{}^i \\
& + \frac{863}{12{,}096} \nabla^4 X_n{}^i + \frac{275}{4{,}032} \nabla^5 X_n{}^i + \frac{33{,}953}{518{,}400} \nabla^6 X_n{}^i \\
& + \frac{8{,}183}{129{,}600} \nabla^7 X_n{}^i + \frac{3{,}250{,}433}{53{,}222{,}400} \nabla^8 X_n{}^i + \frac{4{,}671}{78{,}848} \nabla^9 X_n{}^i \\
& + \frac{301{,}307{,}139{,}941}{5{,}230{,}697{,}472{,}000} \nabla^{10} X_n{}^i + \cdots \Bigg),
\end{aligned}
$$

$$
(2)\qquad {}''X^i{}_{n+1} = 2{}''X_n{}^i - {}''X^i{}_{n-1} + \Delta t^2 X_n{}^i.
$$

Here ∇ is a backward difference operator, i.e.,

$$
\nabla X_n{}^i = X_n{}^i - X^i{}_{n-1}, \quad \nabla^2 X_n{}^i = \nabla X_n{}^i - \nabla X^i{}_{n-1}, \quad \text{etc.}
$$

The $''X_n{}^i$ defined by equation (1) are called "second-summations" [2]. In order to start the integration, $''X_0{}^i$ and $''X^i{}_{-1}$ and all the backward differences for $n = 0$ must be obtained in such a way that the initial conditions are satisfied. One way of doing this is to start the integration using a less accurate but "easy-starting" method with time-step a product of an appropriate negative power of

two times Δt. Then from this solution backward differences can be evaluated and then the second summations for starting Cowell's method can be obtained from (1).

Let m be the order of the last backward difference retained in (1). We shall call $m - 2$ the order of the integration formula (1) and we label the coefficient of the last backward difference retained, c_m. For a computer with large memory we may construct a sequence of instructions for integrating with (1) and (2) in which m and Δt are parameters, i.e., we may compute using any value of m between, say, one and one hundred and any value of Δt. We would like to choose the combination of m and Δt which minimizes the work performed by the computer in obtaining the orbit to a prescribed degree of accuracy. Assuming that most of the work is involved in computing the attractions and not in evaluating the backward difference series we would at first expect to choose Δt close to the limiting value for which the backward difference series in (1) converges. For a circular orbit this limiting value of Δt may be shown to be the time required to traverse an arc of 30° so that we would choose approximately 12 steps per period. However, as will be shown, with 12 steps per period, the integration would become unstable for m greater than 6, so that the radius of convergence of the backward difference series in (1) is not what really determines how large a Δt we should use. It turns out that if m is large enough or if Δt is large enough, spurious solutions of (1) and (2) initiated by rounding errors grow exponentially. In this case we say that the integration procedure is unstable. The spurious solutions are solutions of (1) and (2) which have no analogue in the differential equations of motion, i.e., which vanish as $\Delta t \to 0$. Brouwer [3] has discussed the accumulation of rounding error when we integrate using (1) and (2) neglecting the backward difference series, i.e., when the spurious solutions damp out. Here we wish to estimate the value of Δt at which the integration first becomes unstable as a function of m.

Applying ∇^2 to equation (2) and substituting $\Delta t^2 X_n{}^i$ for $\nabla^{2\,\prime\prime} X^i{}_{n+1}$ we obtain

$$x^i{}_{n+1} = 2x_n{}^i - x^i{}_{n-1} + \Delta t^2 (X_n{}^i + \tfrac{1}{12}\nabla^2 X_n{}^i + \cdots + c_m \nabla^m X_n{}^i). \tag{3}$$

The integration formula (3) is derived by Collatz [4] with coefficients through the fifth order and is attributed by Collatz to Störmer [5]. The solution of (1) and (2) depends on the starting values of $''X_n{}^i$ and $'X_n{}^i$ as well as $m - 1$ values of the coordinates. The solution of (3) depends on $m + 1$ values of the coordinates. Hence the solution of equations (1) and (2) depends on the same number of arbitrary constants as the solution of (3). By choosing the starting conditions properly any solution of (1) and (2) is a solution of (3) and vice versa.

Let us assume that we have a solution to (3) and that then we make a small perturbation in this solution. Let $\epsilon_n{}^i$ be the difference between the coordinates for the new perturbed orbit and the old orbit. Let us suppose that the perturbation is so small that terms in $(\epsilon_n{}^i)^2$ are negligible. Let us assume further that we have only one body in the field of a fixed mass M. Then $X_n{}^i = -\gamma M x_n{}^i / r_n{}^3$. Let us use the notation $X_n{}^{i,j} = \partial X_n{}^i / \partial x_n{}^j$. We obtain equations for $\epsilon_n{}^i$ by subtracting the equations for the perturbed orbit from the equations for the unperturbed orbit and then expanding the attractions for the perturbed orbit about the attrac-

tions for the unperturbed orbit keeping first order terms in $\epsilon_n{}^i$ only. We obtain

$$(4)\quad \epsilon^i{}_{n+1} - 2\epsilon_n{}^i + \epsilon^i{}_{n-1} = \Delta t^2 \sum_{j=1}^{3} [X_n{}^{i,j}\epsilon_n{}^j + \tfrac{1}{12}\nabla^2(X_n{}^{i,j}\epsilon_n{}^j) + \cdots + c_m\nabla^m(X_n{}^{i,j}\epsilon_n{}^j)].$$

The $X_n{}^{i,j}$ are understood to be given functions of the coordinates $x_n{}^i$ of the unperturbed orbit. Let the components of the orthonormal characteristic vectors of the 3×3 matrix $X_n{}^{i,j}$ belonging to characteristic values U_n, V_n, W_n be $u_n{}^i$, $v_n{}^i$, $w_n{}^i$ so that $\sum_j X_n{}^{i,j}u_n{}^j = U_n u_n{}^i$, $\sum_j X_n{}^{i,j}v_n{}^j = V_n v_n{}^i$, and $\sum_j X_n{}^{i,j}w_n{}^j = W_n w_n{}^i$. Written out explicitly, the matrix $X_n{}^{i,j}$ appears as follows:

$$(5)\quad X_n{}^{i,j} = \frac{\gamma M}{r_n{}^3}\begin{bmatrix} -1+3\left(\frac{x_n{}^1}{r_n}\right)^2 & 3\left(\frac{x_n{}^1}{r_n}\right)\left(\frac{x_n{}^2}{r_n}\right) & 3\left(\frac{x_n{}^1}{r_n}\right)\left(\frac{x_n{}^3}{r_n}\right) \\ 3\left(\frac{x_n{}^2}{r_n}\right)\left(\frac{x_n{}^1}{r_n}\right) & -1+3\left(\frac{x_n{}^2}{r_n}\right)^2 & 3\left(\frac{x_n{}^2}{r_n}\right)\left(\frac{x_n{}^3}{r_n}\right) \\ 3\left(\frac{x_n{}^3}{r_n}\right)\left(\frac{x_n{}^1}{r_n}\right) & 3\left(\frac{x_n{}^3}{r_n}\right)\left(\frac{x_n{}^2}{r_n}\right) & -1+3\left(\frac{x_n{}^3}{r_n}\right)^2 \end{bmatrix}.$$

By solving the determinantal equation belonging to this matrix we find that $X_n{}^{i,j}$ has a single root $2\gamma M/r_n{}^3$ and a double root $-\gamma M/r_n{}^3$. Label the single root U_n and its corresponding characteristic vector $u_n{}^i$. We obtain for $u_n{}^i$,

$$(6)\quad u_n{}^i = x_n{}^i/r_n,$$

i.e., $u_n{}^i$ is a unit vector in the direction of the radius vector. The remaining characteristic vectors of $X_n{}^{i,j}$, $v_n{}^i$ and $w_n{}^i$, may be chosen arbitrarily in the plane perpendicular to the radius vector. We expand $\epsilon_n{}^i$ in $u_n{}^i$, $v_n{}^i$, $w_n{}^i$. Let the expansion coefficients be α_n, β_n, γ_n so that

$$(7)\quad \epsilon_n{}^i = \alpha_n u_n{}^i + \beta_n v_n{}^i + \gamma_n w_n{}^i.$$

In order to carry our analysis further we have found it necessary to make the assumption that the orbit is circular. Our results will hold approximately for non-circular orbits in terms of the local radius of curvature provided that the radius of curvature of the orbit does not change too rapidly. Let r, θ be polar coordinates in the plane of motion. Let θ_n be the value of θ corresponding to $x_n{}^i$. It is easily shown that $\Delta\theta^2 = \dfrac{\gamma M}{r^3}\Delta t^2$, where $\Delta\theta$ is the angle between two successive points on the orbit separated in time by Δt. We assume that the unperturbed orbit lies in the x, y plane and that $y = 0$ and $x = r$ when $t = 0$. Then

$$(8)\quad x_n{}^1 = x_n = r\cos n\Delta\theta, \quad x_n{}^2 = y_n = r\sin n\Delta\theta,$$

$$(9)\quad u_n = \begin{pmatrix} \cos n\Delta\theta \\ \sin n\Delta\theta \\ 0 \end{pmatrix}.$$

We choose for v_n^i, w_n^i,

$$v_n = \begin{pmatrix} -\sin n\Delta\theta \\ \cos n\Delta\theta \\ 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

Let S be the rotation matrix for rotating a vector through angle $\Delta\theta$ in the (r, θ) plane. Then

$$S = \begin{pmatrix} \cos\Delta\theta & -\sin\Delta\theta & 0 \\ \sin\Delta\theta & \cos\Delta\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

We have the following relations involving u_n, v_n, w_n, and S:

$$\begin{aligned} u_n &= S^n u_0, \quad v_n = S^n v_0, \quad w_n = S^n w_0, \\ \bar{u}_0 S^n u_0 &= \bar{v}_0 S^n v_0 = \cos n\Delta\theta, \\ \bar{v}_0 S^n u_0 &= -\bar{u}_0 S^n v_0 = \sin n\Delta\theta, \\ \bar{w}_0 S^n u_0 &= \bar{w}_0 S^n v_0 = \bar{u}_0 S^n w_0 = \bar{v}_0 S^n v_0 = 0, \\ \bar{w}_0 S^n w_0 &= 1. \end{aligned} \tag{12}$$

We obtain equations for the expansion coefficients α_n, β_n, γ_n by first substituting (7) into (4), then substituting for u_n, $S^n u_0$; for v_n, $S^n v_0$; and for w_n, $S^n w_0$, and finally multiplying the equations obtained from the preceding steps successively by $\bar{u}_0 S^{-n}$, $\bar{v}_0 S^{-n}$, $\bar{w}_0 S^{-n}$. We then obtain

$$\begin{aligned} \alpha_{n+1}\cos\Delta\theta - 2\alpha_n + \alpha_{n-1}\cos\Delta\theta - \beta_{n+1}\sin\Delta\theta + \beta_{n-1}\sin\Delta\theta \\ = 2\Delta\theta^2 A(\alpha_n) + \Delta\theta^2 B(\beta_n), \end{aligned} \tag{13}$$

$$\begin{aligned} \alpha_{n+1}\sin\Delta\theta - \alpha_{n-1}\sin\Delta\theta + \beta_{n+1}\cos\Delta\theta - 2\beta_n + \beta_{n-1}\cos\Delta\theta \\ = 2\Delta\theta^2 B(\alpha_n) - \Delta\theta^2 A(\beta_n), \end{aligned} \tag{14}$$

$$\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} = -\Delta\theta^2 c(\gamma_n), \tag{15}$$

where A and B are the real and imaginary parts of I and

$$I(\omega_n) = e^{-in\Delta\theta}[\omega_n e^{in\Delta\theta} + \tfrac{1}{12}\nabla^2(\omega_n e^{in\Delta\theta}) + \cdots + c_m\nabla^m(\omega_n e^{in\Delta\theta})], \tag{16}$$

$$c(\gamma_n) = \gamma_n + \tfrac{1}{12}\nabla^2\gamma_n + \cdots + c_m\nabla^m\gamma_n. \tag{17}$$

In (13) and (14) the argument of A and B indicates that α_n or β_n is to be substituted for ω_n in $I(\omega_n)$. We consider first equation (15). Equation (15) has m solutions of the form $\gamma_n = \lambda^n$, λ being obtained by substituting $\gamma_n = \lambda^n$ in (15), multiplying by λ^{m-n-1}, and solving the resulting polynomial of degree m in λ. It may be shown that the two roots λ which approximate the solution of the variational differential equation obtained from (15) when $\Delta\theta \to 0$ lie almost on the unit circle close to $+1$, whereas if $\Delta\theta$ is large enough some roots have negative real parts. As $\Delta\theta$ is increased, one of these roots with negative real part emerges from the unit circle at $\lambda = -1$ on the real axis. The value of $\Delta\theta$ for which $\lambda = -1$

is the largest value of $\Delta\theta$ for which the modulus of all roots is less than or equal to one, and is the largest value of $\Delta\theta$ for which the integration is stable. Let $\Delta\theta_\gamma^m$ be this largest value of $\Delta\theta$ for which the integration formula of order m is stable. Substituting $\lambda = -1$ in (15) we obtain for $\Delta\theta_\gamma^m$,

$$\Delta\theta_\gamma^m = 2\left[1 + \frac{2^2}{12} + \frac{2^3}{12} + \cdots + c_m 2^m\right]^{-1}. \tag{18}$$

Values of $\Delta\theta_\gamma^m$ are tabulated as a function of m in Table 1 at the end of this paper.

Next consider equations (13) and (14). These equations have solutions of the form $\alpha_n = a\lambda^n$, $\beta_n = b\lambda^n$, λ being obtained by substituting $a\lambda^n$, $b\lambda^n$ for α_n, β_n in (13) and (14); multiplying the resulting equations by λ^{m-n-1}, setting the determinant of coefficients of a, b to zero; and solving the resulting determinantal equation of order $2m$. We assume as before that $\lambda = -1$ gives the largest value of $\Delta\theta$ for which all roots have modulus less than one. A proof of this statement for the polynomial equation obtained from (13) and (14) is not easy, and, indeed, the statement may not even be true. However, it is certainly true that if $\lambda < -1$, the integration is unstable, so that the assumption $\lambda = -1$ gives us a bound on $\Delta\theta$ which may not be exceeded. We have had an opportunity to test the stability of various order time-step combinations on a computer and have found that the equations of motion do become unstable at values of $\Delta\theta$ close to those predicted here, for the order time-step combinations tested. This gives us confidence that as $\Delta\theta$ is increased, the first "spurious" root to leave the unit circle must emerge at a point close to minus one. Substituting $\omega_n = \omega(-1)^n$ and making use of the relation

$$\nabla^p(-1)^n e^{in\Delta\theta} = (-1)^n e^{i(n-p/2)\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^p,$$

we obtain

$$I(\omega) = (-1)^n\left[1 + \tfrac{1}{12}e^{-i\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}e^{-i\frac{3}{2}\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^3 + \cdots + c_m e^{-i\frac{m}{2}\Delta\theta}\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]\omega. \tag{19}$$

Then substituting $\alpha_n = a(-1)^n$, $\beta_n = b(-1)^n$ in (13), (14), and multiplying through by $(-1)^n$ we obtain

$$\begin{aligned} -2(1+\cos\Delta\theta)a = {} & 2\Delta\theta^2\left[1 + \tfrac{1}{12}\cos\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}\cos\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3 + \cdots + c_m\cos\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]a \\ & - \Delta\theta^2\left[\tfrac{1}{12}\sin\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}\sin\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3 + \cdots + c_m\sin\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]b, \end{aligned} \tag{20}$$

$$(21)\quad -2(1+\cos\Delta\theta)b = -\Delta\theta^2\left[1+\tfrac{1}{12}\cos\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2 + \tfrac{1}{12}\cos\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3+\cdots+c_m\cos\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]b$$
$$-2\Delta\theta^2\left[\tfrac{1}{12}\sin\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^2+\tfrac{1}{12}\sin\tfrac{3}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^3+\cdots+c_m\sin\frac{m}{2}\Delta\theta\left(2\cos\frac{\Delta\theta}{2}\right)^m\right]a.$$

We solved (20) and (21) for $\Delta\theta$ for several values of m. The results obtained are tabulated under the heading $\Delta\theta^m{}_{\alpha,\beta}$ in Table 1. These solutions were obtained by successive approximation starting with $\Delta\theta_\gamma{}^m$ and then substituting for the trigonometric functions in (20), (21); then solving the determinantal equation for $\Delta\theta^2$, then resubstituting for the trigonometric functions, etc. Two or three iterations were sufficient to obtain $\Delta\theta^m{}_{\alpha,\beta}$ to the number of places given in Table 1.

We note that $\Delta\theta^m{}_{\alpha,\beta}$ is close to $\Delta\theta_\gamma{}^m$. We use $\Delta\theta_\gamma{}^m$ as the limiting value of $\Delta\theta$ for stable integration in subsequent discussion. This approximation is justified since in practice we would not want to choose $\Delta\theta$ close to its limiting value because this would make the rounding error large, and it also is justified because orbits obtained by numerical integration are rarely really circular so that we can only know the limiting value of $\Delta\theta$ approximately. In Table 1 we tabulate $N_m = \dfrac{2\pi}{\Delta\theta_\gamma{}^m}$. N_m is the smallest number of steps per period giving stable integration of order m.

In order to make use of our results we require an estimate of the truncation and rounding error. We assume that Δt has been chosen small enough so that the backward difference series in (3) converges. Let $y_n{}^i$, $Y_n{}^i$ be coordinates and attractions corresponding to the exact solution of the differential system. Then $y_n{}^i$, $Y_n{}^i$ satisfy

$$(22)\qquad \frac{y^i{}_{n+1}-2y_n{}^i+y^i{}_{n+1}}{\Delta t^2}-\sum_{k=0}^{m}c_k\nabla^k Y_n{}^i=\sum_{k=m+1}^{\infty}c_k\nabla^k Y_n{}^i.$$

Our numerical solution $x_n{}^i$ satisfies

$$(23)\qquad \frac{x^i{}_{n+1}-2x_n{}^i+x^i{}_{n+1}}{\Delta t^2}-\sum_{k=0}^{m}c_k\nabla^k X_n{}^i=\rho_n.$$

Here ρ_n is the rounding error incurred at step n. In order to keep as much accuracy as possible without performing extended accuracy multiplication, the term in parenthesis in (3) should be evaluated with such scaling as to give all precision possible using "single-precision" arithmetic, then extra digits should be carried in the summation of the coordinates in (3). Now let $\epsilon_n{}^i = y_n{}^i - x_n{}^i$. Assuming that $\epsilon_n{}^i$ is small, we obtain from (22) and (23) (in manner analogous to the derivation of (4))

$$(24)\qquad \frac{\epsilon^i{}_{n+1}-2\epsilon_n{}^i+\epsilon^i{}_{n+1}}{\Delta t^2}-\sum_{k=0}^{m}\sum_{j=1}^{3}c_k\nabla^k(Y_n{}^{i,j}\epsilon_n{}^j)=-\rho_n+c_{m+1}\nabla^{m+1}Y_n{}^i.$$

In (24) we retain only the first term on the right-hand side of (22). $c_{m+1}\nabla^{m+1}Y_n^i$ is called the local truncation error. For the circular orbit defined by (8)

$$\nabla^{m+1}Y_n^1 = -\frac{\gamma M}{r^2}\left(2\sin\frac{\Delta\theta}{2}\right)^{m+1}\cos(\omega t_n - \varphi_m),$$

$$\nabla^{m+1}Y_n^2 = -\frac{\gamma M}{r^2}\left(2\sin\frac{\Delta\theta}{2}\right)^{m+1}\sin(\omega t_n - \varphi_m), \tag{25}$$

$$\nabla^{m+1}Y_n^3 = 0,$$

where $\omega = \sqrt{\dfrac{\gamma M}{r^3}}$ and φ_m equals $\dfrac{m+1}{2}(\Delta\theta - \pi)$. We let $\Delta_{m+1} = c_{m+1}\left(2\sin\dfrac{\Delta\theta}{2}\right)^{m+1}$.

As can be seen by examining equations (22) and (23), if the problem is scaled so that $(Y_n^i)_{\max} = \gamma M/r^2$ occupies the full number size of the computer, then the local truncation error is the same order as the rounding error when Δ_{m+1} is the same order of magnitude as ρ_n. This assumes that extra accuracy is carried in the coordinates as mentioned previously. Δ_{m+1} is tabulated in Table 1.

We may now apply our results and choose an appropriate time-step size and order, m, for various particular cases. For example, suppose we wish to integrate for orbits of the planets on a computer with a 14 decimal-digit number size and we wish to obtain the orbits with all the precision of which the machine is capable without performing extended accuracy operations other than extended accuracy summations. We see from Table 1 that we are going to need between 60 and 70 steps with $m = 12$ to keep the truncation error small enough and that then we are too close to the unstable value of $\Delta\theta$, so that we must use more steps, say 100 steps per period. But then the truncation error is sufficiently small with $m = 10$, 11, or 12. $m = 11$ and 100 steps per period would appear to be a good compromise. In the integrations of the orbits of the five outer planets, Eckert, Brouwer, and Clemence [2] used $m = 11$ and they had approximately 113 steps per period in Jupiter, the planet with the smallest period. Hence this theoretical analysis gives results in good agreement with what has been found appropriate in practice.

Many computers have a 10 decimal-digit word size. On these $m = 10$ with 60 steps per revolution would appear to be a good choice.

We see that we would not expect to use very high orders ($m > 14$) under any practical circumstances. An extreme case would be one in which we wished to follow a very large number of revolutions, say 10^{10} revolutions. We would then wish to perform extended accuracy computations on, say, a 14 digit machine thus obtaining effectively 28 decimal-digit number size. In this case, to keep the local truncation error below 10^{-28} and to be assured of stability we would want to choose about 180 time-steps per period and $m = 13$. Assuming we could perform the computations for one time-step in a millisecond (this is probably somewhat faster than we could expect from any computer today) it would require 56 years to complete the computation. Hence even in an extreme case m is less than 14.

Of course we do not always wish to integrate obtaining the maximum precision of which a particular computer with its given number size is capable. Instead, we often wish the error in the orbit to be less than a given bound over the total

interval of the integration. In order to be assured of this it is necessary to estimate the accumulated truncation error and the accumulated rounding error. From our prescribed bound in the maximum allowable accumulated truncation error we obtain the maximum allowable Δt for each order m. Then for least work by the computer we choose the $(m, \Delta t)$ combination which gives the largest value of Δt for which the integration is stable according to Table 1. We obtain the minimum allowable number size from our estimate of the accumulated rounding error. Then we have obtained the maximum Δt and minimum number size which may be used to solve our problem using Cowell's method.

Let us estimate the accumulated truncation error. It is easily seen that equation (24) approximates the differential system

$$d^2\epsilon^i/dt^2 - \sum_{j=1}^{3} Y^{i,j}\epsilon^j = -\rho + c_{m+1}\nabla^{m+1}Y^i \tag{26}$$

where ϵ^i, $Y^{i,j}$, ρ, and $c_{m+1}\nabla^m Y^i$ now stand for continuous quantities. For the circular orbit we can obtain exactly a particular solution of equation (26) corresponding to the local truncation error. It is

$$\epsilon^i \text{ (truncation)} = r\Delta_{m+1}[(\tfrac{1}{3}\cos\varphi_m + 2\theta\sin\varphi_m)u^i - \tfrac{3}{2}\theta^2\sin\varphi_m v^i]. \tag{27}$$

Here φ_m is the phase factor defined following equation (25) and u^i and v^i are defined in equations (9) and (10) except that here $n\Delta\theta$ is replaced by a continuous variable $\theta = \omega t$. We see that after a large number, P, of periods, the last term in (27) dominates and the amplitude of the truncation error approaches $|6\pi^2 \sin\varphi_m P^2 \Delta_{m+1} r|$. The way the rounding error accumulates can also be investigated starting with (26). One is led to the same result obtained by Brouwer [3], namely that after n steps the root-mean-square rounding error increases by a factor of $n^{\frac{1}{2}}$. The method used here to estimate the accumulated truncation error is similar to that discussed by Rademacher [6].

TABLE 1. *Values of $\Delta\theta_\gamma{}^m$, $\Delta\theta^m{}_{\alpha,\beta}$, N_m, Δ_{m+1} for $m = 6(1)14$*

m	$\Delta\theta_\gamma{}^m$	$\Delta\theta^m{}_{\alpha,\beta}$	N_m	Δ_{m+1}	m	$\Delta\theta_\gamma{}^m$	$\Delta\theta^m{}_{\alpha,\beta}$	N_m	Δ_{m+1}
6	.6252	.629	10.05	10^{-3}	11	.1260		49.86	10^{-12}
7	.4593		13.68	10^{-4}	12	.0905	.0893	69.39	10^{-15}
8	.3346	.3211	18.78	10^{-6}	13	.0650		96.70	10^{-18}
9	.2424		25.92	10^{-8}	14	.0466		134.88	10^{-21}
10	.1750	.1701	35.90	10^{-10}					

Note: Values listed for m = 13, 14, and Δ_{13} are based on estimated values of C_{13}, C_{14}, and C_{15}.

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Optimum Quadrature Formulas In s Dimensions

Although numerical procedures for functions of more than one variable are of considerable practical importance, they have received relatively little attention. So far as numerical integration is concerned, aside from successive application of appropriate one-dimensional results, as discussed in standard texts, and a few isolated special results for two dimensions, there has been little systematic study of the problem. Tyler [1] has collected a set of formulas of up to seventh degree for integration over rectangular regions in the plane, and up to fifth degree in three dimensions, and also gives a $(2s + 1)$-point third-degree formula for s dimensions. Hammer, Marlowe, Stroud, and Wymore [2, 3, 4] have pointed out useful methods of extending available results for spaces of a given dimension to higher dimensions, and to regions which are related to the given region by affine transformations. They have also developed a number of formulas for simplexes, cones, and spheres in s dimensions.

It is the purpose of the present paper to describe a general approach to the problem which may often yield useful results and to use it to derive formulas in s dimensions using $(s + 1)$ points for second-degree accuracy, and $2s$ points for third-degree accuracy. This method is applicable to any region, although its advantages are most pronounced for regions with a center of symmetry, and in our detailed calculations we will limit ourselves to a hypercube with center at the origin. We will be concerned with formulas of the form

$$(1)\quad \int_S \cdots \int \varphi(x^{(1)}, \cdots, x^{(s)})dx^{(1)} \cdots dx^{(s)} = \sum_{i=1}^{m} c_i\varphi(x_i^{(1)}, \cdots, x_i^{(s)}) - R\varphi$$

where $R\varphi$ denotes the error of the formula for a particular function φ. A formula will be said to be of degree n if $R\varphi$ is zero whenever φ is a polynomial of degree n or less in the s variables $x^{(j)}$.

If a formula is of degree n, the c_i and $x_i^{(j)}$ must satisfy the set of equations

$$(2)\quad \sum_{i=1}^{m} c_i \prod_{j=1}^{s} (x_i^{(j)})^{n_j} = \int_S \cdots \int \prod_{j=1}^{s} (x^{(j)})^{n_j} dx^{(j)} \equiv I_{n_1, \cdots, n_s}$$

for all sets of n_j with sum equal to or less than n. An nth degree formula in s dimensions involves $(n+s)!/n!s!$ such nonlinear inhomogeneous algebraic equations, and each solution to such a set leads to an acceptable nth degree formula. Unfortunately, the task of solving systems of nonlinear algebraic equations is a formidable one, and general solutions have only been obtained in the simplest cases, although special methods have led to single solutions for more complex cases [1].

An instructive formulation of the problem can be obtained by a change in viewpoint. Let us define a set of $m \times m$ diagonal matrices by

$$\mathbf{G} \equiv [\sqrt{c_i}\,\delta_{hi}] \tag{3}$$

$$\mathbf{X}^{(j)} \equiv [x_i^{(j)}\delta_{hi}]. \tag{4}$$

In terms of these matrices, (2) becomes

$$\operatorname{tr}\left\{\mathbf{G}\prod_{j=1}^{s}(\mathbf{X}^{(j)})^{n_j}\mathbf{G}\right\} = I_{n_1,\ldots,n_s}. \tag{5}$$

Although the solution of a set of matrix equations such as (5) is generally no easier than the solution of a set of nonlinear algebraic equations such as (2), this formulation does emphasize certain considerations which are generally used intuitively in solving (2). In the first place, the fact that the validity of a formula of the form of equation (1) is independent of the numbering of the points is reflected in the fact that the trace of a diagonal matrix is unaffected by interchange of diagonal elements. If the region S is symmetric with respect to change of sign of one of the variables, this fact must be reflected by an invariance of (5) to a change of sign of $\mathbf{X}^{(j)}$. If the region S is symmetric with respect to permutation of certain variables, $I_{n_1,\ldots,n_s}$ must be invariant to permutation of the corresponding n_j.

Let us now specialize this to the case of hypercubes with edge length two, and center at the origin. In this case, the $I_{n_1,\ldots,n_s}$ are independent of the order of the subscripts, and

$$I_{n_1,\ldots,n_s} = \begin{cases} 0 & \text{if at least one } n_j \text{ is odd} \\ \dfrac{2^s}{\prod_{j=1}^{s}(n_j+1)} & \text{if no } n_j \text{ is odd.} \end{cases} \tag{6}$$

For a second degree formula, therefore, we have the equations

$$\operatorname{tr}\{\mathbf{G}\mathbf{G}\} = 2^s \tag{7}$$

$$\operatorname{tr}\{\mathbf{G}\mathbf{X}^{(j)}\mathbf{G}\} = 0 \tag{8}$$

$$\operatorname{tr}\{\mathbf{G}\mathbf{X}^{(j)}\mathbf{X}^{(k)}\mathbf{G}\} = \frac{2^s}{3}\,\delta_{jk}. \tag{9}$$

These equations in the traces may be converted to vector equations if we introduce the m-dimensional column vector with all elements unity, ϵ, and its transpose, ϵ^T. Then if we define the vectors $\xi, \xi_j, \xi_{jk}, \cdots$, by $\xi = \mathbf{G}\epsilon$, $\xi_j = \mathbf{X}^{(j)}\mathbf{G}\epsilon$, $\xi_{jk} = \mathbf{X}^{(j)}\mathbf{X}^{(k)}\mathbf{G}\epsilon$ and so on, (7), (8) and (9) become

$$\epsilon^T\mathbf{G}\mathbf{G}\epsilon = (\mathbf{G}\epsilon)^T(\mathbf{G}\epsilon) = \xi^T\xi = 2^s \tag{10}$$

$$\epsilon^T\mathbf{G}\mathbf{X}^{(j)}\mathbf{G}\epsilon = (\mathbf{G}\epsilon)^T(\mathbf{X}^{(j)}\mathbf{G}\epsilon) = \xi^T\xi_j = 0 \tag{11}$$

$$\begin{aligned}\epsilon^T\mathbf{G}\mathbf{X}^{(j)}\mathbf{X}^{(k)}\mathbf{G}\epsilon &= (\mathbf{X}^{(j)}\mathbf{G}\epsilon)^T(\mathbf{X}^{(k)}\mathbf{G}\epsilon) = \xi_j{}^T\xi_k \\ &= (\mathbf{X}^{(j)}\mathbf{X}^{(k)}\mathbf{G}\epsilon)^T(\mathbf{G}\epsilon) = \xi^T{}_{jk}\xi = \frac{2^s}{3}\,\delta_{jk}.\end{aligned} \tag{12}$$

These, however, will be recognized as merely orthogonality relations among $(s + 1)$ vectors, $\xi, \xi_1, \cdots, \xi_s$ and normalization requirements that $|\xi|^2 = 2^s$, $|\xi_j|^2 = 2^s/3$. Now $(s + 1)$ orthogonal vectors span a vector space of dimension $(s + 1)$ and this space must be a subspace of the vector space of dimension m consisting of all m-dimensional vectors. Thus, a second degree formula of the form of (1) can be obtained with $m = s + 1$, and for any higher value of m.

Our argument has also furnished an explicit algorithm for constructing examples of such formulas by orthogonalizing any linearly independent set of $(s + 1)(s + 1)$-dimensional vectors, and applying the proper normalization conditions. For example, if we orthogonalize the set, $(1, 1, 1)$, $(3, -\sqrt{3}\tan\vartheta, \sqrt{3}\tan\vartheta)$ and $(3\sqrt{3}\tan\vartheta, 0, 0)$, in that order, we find

$$\xi = (2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3}) \tag{13}$$

$$\xi_1 = \left(\frac{2\sqrt{2}}{3}\cos\vartheta, \frac{2\sqrt{2}}{3}\cos\left(\vartheta + \frac{2\pi}{3}\right), \frac{2\sqrt{2}}{3}\cos\left(\vartheta + \frac{4\pi}{3}\right)\right) \tag{14}$$

$$\xi_2 = \left(\left(\frac{2\sqrt{2}}{3}\right)\sin\vartheta, \frac{2\sqrt{2}}{3}\sin\left(\vartheta + \frac{2\pi}{3}\right), \frac{2\sqrt{2}}{3}\sin\left(\vartheta + \frac{4\pi}{3}\right)\right) \tag{15}$$

so that our integration formula becomes

$$\begin{aligned}&\int_{-1}^{1}\int_{-1}^{1}\varphi(s, t)\,ds\,dt \\ &= \frac{4}{3}\left\{\varphi\left(\sqrt{\tfrac{2}{3}}\cos\vartheta, \sqrt{\tfrac{2}{3}}\sin\vartheta\right) + \varphi\left(\sqrt{\tfrac{2}{3}}\cos\left(\vartheta + \frac{2\pi}{3}\right), \sqrt{\tfrac{2}{3}}\sin\left(\vartheta + \frac{2\pi}{3}\right)\right)\right. \\ &\qquad \left. + \varphi\left(\sqrt{\tfrac{2}{3}}\cos\left(\vartheta + \frac{4\pi}{3}\right), \sqrt{\tfrac{2}{3}}\sin\left(\vartheta + \frac{4\pi}{3}\right)\right)\right\} - R_2\varphi\end{aligned} \tag{16}$$

where $R_2\varphi$ vanishes if φ is a polynomial of degree 2 or less. The parameter ϑ is arbitrary, although because of the periodicity of the trigonometric functions it may be taken to lie in the range $-\frac{\pi}{3} \leq \vartheta < \frac{\pi}{3}$. The vertices of any equilateral

triangle inscribed in a circle of radius $\sqrt{\tfrac{2}{3}}$ are thus seen to afford a satisfactory set of second-degree integration points.

For higher degree formulas, the orthogonality conditions no longer suffice to define the minimal solution completely, but they still afford a substantial simplification, as can be seen from the following discussion of third-degree formulas. For these, we have, along with (10), (11) and (12), the condition

$$\text{(17)} \qquad \epsilon^T \mathbf{G}\mathbf{X}^{(j)}\mathbf{X}^{(k)}\mathbf{X}^{(l)}\mathbf{G}\epsilon = (\mathbf{X}^{(j)}\mathbf{X}^{(k)}\mathbf{G}\epsilon)^T(\mathbf{X}^{(l)}\mathbf{G}\epsilon) = \xi^T{}_{jk}\xi_l = 0.$$

Thus, we must consider the $s(s+1)/2$ new vectors ξ_{jk} in addition to ξ and the ξ_j. These new vectors fall into two classes: the s ξ_{jj} are orthogonal to every ξ_l, but not to ξ, while the $s(s-1)/2$ $\xi_{jk}(j \neq k)$ are orthogonal to both sets, or else are null vectors.

Let us consider first the case where all the ξ_{jk} are null vectors. This implies that unless one or more elements of ξ is zero, in which case the basic integration formula includes redundant points with zero weight, only one of the $\mathbf{X}^{(j)}$ can have any given element different from zero. The ξ_{jj} cannot be null vectors, in view of (12), while from (11), unless the c_i differ in sign, $\mathbf{X}^{(j)}$ must include elements of both signs, and thus at least two non-zero elements. We therefore conclude that the dimension of $\mathbf{X}^{(j)}$ must be at least $2s$.

For an equally-weighted formula, $\mathbf{G}$ is a scalar matrix, which we may write as $g\mathbf{E}$, where $\mathbf{E}$ is the identity. From (7), for a $2s$-point formula, g^2 must have the value $2^{s-1}/s$. For the minimum number of points, $\mathbf{X}^{(j)}$ will have but two non-zero elements of opposite sign, and, from (8), of equal magnitude, and these may be arranged in order such that

$$\text{(18)} \qquad (\mathbf{X}^{(j)})_{hi} = x^{(j)}(\delta_{hi}[\delta_{i,2j-1} - \delta_{i,2j}])$$

a diagonal matrix with the $(2j-1)$th element equal to $x^{(j)}$ and the $2j$th to $-x^{(j)}$. From (9), it follows that

$$\text{(19)} \qquad 2x^{(j)2}g^2 = 2^s/3$$

so that for all j,

$$\text{(20)} \qquad x^{(j)} = \sqrt{s/3}$$

and we have the family of $2s$-point third-degree integration formulas

$$\text{(21)} \qquad \int_{-1}^{1}\cdots\int_{-1}^{1} \varphi(x^{(1)}, \cdots, x^{(s)})dx^{(1)} \cdots dx^{(s)} = \frac{2^{s-1}}{s}\sum_{i=1}^{2s} \varphi(x_i^{(1)}, \cdots, x_i^{(s)}) + R_3\varphi$$

where

$$\text{(22)} \qquad x_i^{(j)} = \sqrt{\frac{s}{3}}\,(\delta_{i,2j-1} - \delta_{i,2j}).$$

The case for s equal to three is included in Tyler's collection [1]. For s greater than three, these formulas have the drawback that they depend upon values of

the function outside the range of integration, and thus may have larger remainder terms than are desirable.

If one or more of the ξ_{jk} is non-null, there must be $s + 2$ or more orthogonal vectors, and the minimum value of m must be at least $s + 2$. This lower bound can be attained for s equal to two. In this case, we have a basis of four orthogonal vectors, ξ, ξ_1, ξ_2, and ξ_{12}. If a four-point formula exists, these vectors completely span the space, and we can express ξ_{11} and ξ_{22} as linear combinations of ξ and ξ_{12},

$$\xi_{jj} = a_j\xi + b_j\xi_{12} \tag{23}$$

since ξ_{11} and ξ_{22} are orthogonal to ξ_1 and ξ_2. If we multiply both sides of (23) by ξ^T, it follows from (12) that a_j must be $\frac{1}{3}$. Hence, introducing the definition of ξ_{12}, and rearranging,

$$\mathbf{X}^{(j)}(\mathbf{X}^{(j)} - b_j\mathbf{X}^{(k)})\xi = \tfrac{1}{3}\xi. \tag{24}$$

Thus ξ is an eigenvector, and $\frac{1}{3}$ an eigenvalue of the two matrices, $\mathbf{X}^{(1)}(\mathbf{X}^{(1)} - b_1\mathbf{X}^{(2)})$ and $\mathbf{X}^{(2)}(\mathbf{X}^{(2)} - b_2\mathbf{X}^{(1)})$. Since these matrices are diagonal, all the elements corresponding to non-zero elements of ξ must be equal to $\frac{1}{3}$. In particular, for equally-weighted four-point formulas, the diagonal elements of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ must be the roots of the two equations

$$x^{(1)^2} - b_1x^{(1)}x^{(2)} = \tfrac{1}{3}, \quad x^{(2)^2} - b_2x^{(1)}x^{(2)} = \tfrac{1}{3}, \tag{25}$$

which become

$$x^{(2)} = (x^{(1)^2} - \tfrac{1}{3})/b_1x^{(1)}, \tag{26}$$

and

$$x^{(1)^4} - \frac{2 - b_1b_2 + b_1^2}{3(1 - b_1b_2)}\,x^{(1)^2} + \frac{1}{9(1 - b_1b_2)} = 0. \tag{27}$$

Now the sum of the squares of the four roots of (27) is equal to -2 times the second term, and if these roots are to satisfy (9)

$$2(2 - b_1b_2 + b_1^2) = 3(1 - b_1b_2)\tfrac{4}{3} \tag{28}$$

so that

$$b_1 = -\,b_2 \equiv b. \tag{29}$$

Accordingly,

$$x^{(1)^4} - \tfrac{2}{3}x^{(1)^2} + \frac{1}{9(1 + b^2)} = 0 \tag{30}$$

$$x^{(1)} = \pm\left\{\frac{1}{3}\left(1 \pm \sqrt{\frac{b^2}{1 + b^2}}\right)\right\}^{\frac{1}{2}} \tag{31}$$

$$x^{(2)} = \pm \left\{ \frac{1}{3}\left(1 \mp \sqrt{\frac{b^2}{1+b^2}}\right) \right\}^{\frac{1}{2}}. \tag{32}$$

Thus, again we find that the integration points lie on a circle of radius $\sqrt{2/3}$, and that there is a whole family of four-point third-degree cubature formulas, corresponding to the corners of squares inscribed in this circle. In three dimensions, it can be shown that a five-point third-degree formula is impossible, but that the vertices of any regular octahedron inscribed in a sphere of unit radius are satisfactory points for an equally-weighted quadrature formula. Even with matrix notation, the calculations for spaces of higher dimension, or for formulas of higher degree become extremely tedious. However, it seems likely that, as was observed for second- and third-degree formulas, the minimum number of points necessary for a quadrature formula of a given dimension, s, will continue to be of the order of some power of s, and not of the sth power of the number of points for a one-dimensional formula.

It should be noticed that the application of the matrix-vector formulation is not limited to the hypercubical regions with constant weighting function which have been discussed in detail. It is equally helpful in deriving multidimensional analogues of the Gauss-Laguerre and Gauss-Hermite one-dimensional formulas, although the corresponding relations among the vectors are no longer simple orthogonality conditions, but require less obvious methods of solution.

Because of the variety of formulas discussed, the problem of the truncation error for formulas of various degrees and dimension cannot be considered in detail. The more familiar methods for obtaining remainders in one dimension cannot, unfortunately, be extended to multidimensional problems, and Sard's extension [5] of Peano's theorem is one of the very few useful bases for an error estimate. Even so, the expressions are complicated, and depend upon several partial derivatives, thus requiring quite detailed information about the behavior of the integrand, which may often be unavailable.

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TECHNICAL NOTES AND SHORT PAPERS

A Method to Investigate Primality

The method determines the smallest odd prime factor of a number N by testing the remainders left after division by the successive odd numbers 3, 5, $\cdots$ $f_m - 2$, f_m: here, f_m is the largest odd number not exceeding $N^{\frac{1}{2}}$. If none of these remainders vanishes, N is a prime number.

Let f be one of the odd trial divisors. Remainder r_0 and quotient q_0 are defined by the relations

$$N = r_0 + fq_0, \quad 0 \leq r_0 < f.$$

Now q_0 is divided by $f + 2$, giving

$$q_0 = r_1 + (f + 2)q_1, \quad 0 \leq r_1 < f + 2.$$

Then q_1 is divided by $(f + 4)$, etc., and this process is continued till a quotient (q_n, say) equal to zero is found; r_n is the last remainder in the sequence unequal to zero. After elimination of the q_i we get the relations

$$(1) \quad N = r_0 + fr_1 + f(f+2)r_2 + f(f+2)(f+4)r_3 + \cdots + f(f+2) \cdots (f+2n-2)r_n$$

and

$$(2) \qquad 0 \leq r_i < f + 2i.$$

Once the sequence r_i is known for a given value of f, it is easy to compute the corresponding sequence r_i^*, defined by the relations (1) and (2) with respect to $f^* = f + 2$, as they are expressed in terms of the r_i by the recurrence relations

$$(3) \quad b_0 = 0, \quad r_i^* = r_i - 2(i+1)r_{i+1} - b_i + (f^* + 2i)b_{i+1}, \quad (i = 0, 1, \cdots, n).$$

The relation corresponding to (1) is satisfied for arbitrary values of the numbers b_i with $i \geq 1$; they are fixed, however, by the relations corresponding to (2)

$$(2^*) \qquad 0 \leq r_i^* < f^* + 2i.$$

On account of the inequalities (2) and (2*)—and $b_0 = 0$—the b_i satisfy the inequalities

$$(4) \qquad 0 \leq b_i \leq 2i.$$

We have chosen $b_0 = 0$. Then the relations (3) and (2*) with $i = 0$ determine r_0^* and b_1; once b_1 is known, (3) and (2*) with $i = 1$ determine r_1^* and b_2, etc. The process is easily programmed.

As $r_{n+1} = 0$, and the inequalities (2*) with $i = n$ are always satisfied with $b_{n+1} = 0$, the process terminates with

$$r_n^* = r_n - b_n.$$

As soon as $r_n^* = 0$ is found—in that case it can be proved that $r^*_{n-1} \neq 0$—the index n, marking the last $r_i \neq 0$ in the sequence, is lowered by 1.

In order to find the smallest odd prime factor of N, the r_i defined by (2) and (3) and $f = 3$ are computed. Here the only divisions in the process are carried out. At the same time the initial value of n is found. If N is large, this value may be considerable: for instance $n = 11$ is found for $N \approx 10^{12}$. The amount of work involved in each step is roughly proportional to n^2. Fortunately large initial values of n decrease very rapidly. As soon as $f \cdot (f + 2) \cdot (f + 4) > N$, n takes the value 2. This is its minimum value: when $r_n^* = 0$ with $n = 2$ is found, $(f^* + 2)^2 > N$ holds and N is a prime number. (If not, we should have found an $r_0 = 0$ earlier and should have stopped there.)

The process still may be speeded up. Let b_n' be the minimum of b_n for fixed n up till a certain moment: then it can be shown that the next b_n satisfies

$$b_n \leq b_n' + 1. \tag{5}$$

Let us apply this to the last stage $n = 2$. According to (4) b_2 satisfies $0 \leq b_2 \leq 4$. According to (5), however, the only possible values for b_2 are 0 and 1 as soon as a value $b_2 = 0$ once has been found. This is bound to happen for f ranging (roughly) from $(4N)^{\frac{1}{3}}$ to $(8N)^{\frac{1}{3}}$. In the case $b_2 = 0$ it is apparently unnecessary to test whether $r_2 = 0$ is reached. (If $N \geq 144$, the case $b_n = 0$ with $n = 2$ occurs, before $r_n^* = 0$ with $n = 2$ is found; prime numbers are then always detected in this last stage.)

The less efficient steps of the process for large n (i.e., small f) could be avoided by carrying out divisions for small values of f (see Alway [1]). However we strongly advise against doing this.

If the process described above is started at $f = 3$, the *whole* computation can be checked at the end by inserting the final values of f and r_i into (1). As all the intermediate results are used in the computation, this check seems satisfactory.

If a double-length number N is to be investigated, another argument can be added: division of N by small f may give a double-length quotient, i.e., two divisions (and two multiplications to check) are needed for each f. In our case only part of the initial n divisions are double-length divisions.

The process described above has been programmed for the ARMAC (Automatische Rekenmachine van het Mathematisch Centrum). The speed of this machine is about 2400 operations per second. A twelve decimal number was identified as the square of a prime in less than 23 minutes.

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Equally-Weighted Quadrature Formulas for Inversion Integrals

In a previous article [1] the author considered Gaussian-type quadrature formulas for the numerical evaluation of inversion integrals $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p} F(p)dp$, where an n-point formula was exact whenever $F(p)$ was a polynomial of degree $(2n-1)$ in $1/p$. In this present note we consider equally weighted (i.e., Chebyshev type) quadrature formulas of the form

$$(1) \qquad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p} F(p)dp = \frac{1}{n}\sum_{j=1}^{n} F(p_j),$$

where (1) is exact whenever $F(p)$ is any polynomial of degree n in $1/p$. The analogous set of equally weighted quadrature formulas for evaluating infinite integrals that are direct Laplace transforms has already been considered in one of the author's earlier papers [2]. Similar to the derivation given there, it is easily seen here that the well-known relation

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^p}{p}\left(\frac{1}{p}\right)^r dp = \frac{1}{r!},$$

by choosing $r = 0$, establishes the factor $1/n$ outside the summation in (1), and the choice of $r = 1, 2, \cdots, n$ establishes the following necessary and sufficient conditions on p_j, in order that (1) should hold whenever $F(p)$ is an arbitrary nth degree polynomial in $1/p$

$$(2) \qquad \sum_{j=1}^{n}\left(\frac{1}{p_j}\right)^r = \frac{n}{r!}, \quad r = 1, 2, \cdots, n.$$

Following the usual methods [2], one determines for the n-point case of (1) the coefficients of the polynomials $\phi_n(z)$ whose zeros z_j are the reciprocals of the required points p_j. Thus from $\phi_n(z) = z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_{n-1}z + a_n$, the coefficients a_k are found successively from

$$(3) \quad ka_k + a_{k-1}\sum_{j=1}^{n} z_j + a_{k-2}\sum_{j=1}^{n} z_j^2 + \cdots + a_1\sum_{j=1}^{n} z_j^{k-1} + \sum_{j=1}^{n} z_j^k = 0,$$
$$k = 1, 2, \cdots, n.$$

Below are the general formulas for the first ten coefficients $a_1, a_2, \cdots, a_{10}$ for any n, having meaning, of course, only for an a_k where $k \leq n$

$$a_1 = -n,$$

$$a_2 = \frac{n^2}{2} - \frac{n}{4},$$

$$a_3 = -\frac{n^3}{6} + \frac{n^2}{4} - \frac{n}{18},$$

$$a_4 = \frac{n^4}{24} - \frac{n^3}{8} + \frac{25}{288}n^2 - \frac{n}{96},$$

$$a_5 = -\frac{n^5}{120} + \frac{n^4}{24} - \frac{17}{288}n^3 + \frac{7}{288}n^2 - \frac{n}{600},$$

$$a_6 = \frac{n^6}{720} - \frac{n^5}{96} + \frac{43}{1728}n^4 - \frac{25}{1152}n^3 + \frac{1507}{2\,59200}n^2 - \frac{n}{4320},$$

$$a_7 = -\frac{n^7}{5040} + \frac{n^6}{480} - \frac{13}{1728}n^5 + \frac{13}{1152}n^4 - \frac{1741}{2\,59200}n^3 + \frac{53}{43200}n^2 - \frac{n}{35280},$$

$$a_8 = \frac{n^8}{40320} - \frac{n^7}{2880} + \frac{61}{34560}n^6 - \frac{7}{1728}n^5 + \frac{17627}{41\,47200}n^4 - \frac{3779}{20\,73600}n^3 + \frac{15787}{677\,37600}n^2 - \frac{n}{3\,22560},$$

$$a_9 = -\frac{n^9}{3\,62880} + \frac{n^8}{20160} - \frac{7}{20736}n^7 + \frac{19}{17280}n^6 - \frac{22289}{124\,41600}n^5 + \frac{2887}{20\,73600}n^4 - \frac{4\,87471}{10973\,49120}n^3 + \frac{12317}{3048\,19200}n^2 - \frac{n}{32\,65920},$$

$$a_{10} = \frac{n^{10}}{36\,28800} - \frac{n^9}{1\,61280} + \frac{79}{14\,51520}n^8 - \frac{11}{46080}n^7 + \frac{69353}{1244\,16000}n^6 - \frac{11347}{165\,88800}n^5 + \frac{22\,37339}{54867\,45600}n^4 - \frac{26791}{2709\,50400}n^3 + \frac{5\,90383}{9\,14457\,60000}n^2 - \frac{n}{362\,88000}.$$

The first ten polynomials $\phi_n(z)$ are

$$\phi_1(z) = z - 1,$$

$$\phi_2(z) = z^2 - 2z + \frac{3}{2},$$

$$\phi_3(z) = z^3 - 3z^2 + \frac{15}{4}z - \frac{29}{12},$$

$$\phi_4(z) = z^4 - 4z^3 + 7z^2 - \frac{62}{9}z + \frac{289}{72},$$

$$\phi_5(z) = z^5 - 5z^4 + \frac{45}{4}z^3 - \frac{535}{36}z^2 + \frac{1805}{144}z - \frac{1627}{240},$$

$$\phi_6(z) = z^6 - 6z^5 + \frac{33}{2}z^4 - \frac{82}{3}z^3 + \frac{481}{16}z^2 - \frac{4537}{200}z + \frac{27769}{2400},$$

$$\phi_7(z) = z^7 - 7z^6 + \frac{91}{4}z^5 - \frac{1631}{36}z^4 + \frac{4417}{72}z^3 - \frac{1\,06351}{1800}z^2 + \frac{53\,02619}{1\,29600}z - \frac{180\,44381}{9\,07200},$$

$$\phi_8(z) = z^8 - 8z^7 + 30z^6 - \frac{628}{9}z^5 + \frac{4037}{36}z^4 - \frac{3277}{25}z^3 + \frac{9\,22919}{8100}z^2 - \frac{29\,22187}{39690}z + \frac{1455\,11171}{42\,33600},$$

$$\phi_9(z) = z^9 - 9z^8 + \frac{153}{4}z^7 - \frac{407}{4}z^6 + \frac{3027}{16}z^5 - \frac{1\,03911}{400}z^4 + \frac{6\,50239}{2400}z^3 - \frac{84\,99571}{39200}z^2 + \frac{414\,78457}{3\,13600}z - \frac{15146\,11753}{254\,01600},$$

$$\phi_{10}(z) = z^{10} - 10z^9 + \frac{95}{2}z^8 - \frac{1280}{9}z^7 + \frac{43235}{144}z^6 - \frac{1\,70381}{360}z^5 + \frac{14\,90251}{2592}z^4 - \frac{173\,63761}{31752}z^3 + \frac{4151\,58089}{10\,16064}z^2 - \frac{32552\,25203}{137\,16864}z + \frac{14\,23249\,22009}{13716\,86400}.$$

The values of p_j and the zeros $z_j \equiv 1/p_j$ of the above polynomials $\phi_n(z)$, for $n = 1(1)10$, are given in Table 1.

The zeros $z_j \equiv 1/p_j$ of $\phi_n(z)$ were calculated for $n = 3, 4, \cdots, 10$, by first obtaining an initial approximation, using a procedure that had been employed upon the Univac Scientific Computer (ERA 1103) at the Convair Digital Computing Laboratory. The initial approximations to the complex zeros were then used to construct approximate real quadratic factors, which were refined by Bairstow's method (Milne [3], Olver [4]), using only a desk calculator. The initial approximations to the real zeros were refined by Newton's method. All factors of $\phi_n(z)$ were checked by four different formulas (see [4], p. 414). Also a final functional check was performed upon the values of $1/p_j$ by substituting into equation (2) for $n = 1(1)10$, and $r = 1(1)n$.

The 8-decimal values of p_j and $1/p_j$ in Table 1 are guaranteed as far as the seventh decimal place. But they are believed to be correct to within around two units in the eighth decimal place for $n = 1(1)7$ and have a high probability of being correct to within several units in the eighth decimal place for $n = 8(1)10$.

TABLE 1. p_j *and* $1/p_j$

n	j	p_j	$1/p_j$
1	1	1.00000 000 + .00000 000i	1.00000 000 + .00000 000i
2	1, 2	.66666 667 ± .47140 452i	1.00000 000 ∓ .70710 678i
3	1, 2	.46343 318 ± .66891 655i	.69981 792 ∓ 1.01011 279i
	3	.62485 778 + .00000 000i	1.60036 417 + .00000 000i
4	1, 2	.31209 699 ± .78442 870i	.43788 772 ∓ 1.10059 277i
	3, 4	.54603 449 ± .22670 497i	1.56211 228 ∓ .64856 456i
5	1, 2	.19029 304 ± .86260 499i	.24387 201 ∓ 1.10548 034i
	3, 4	.46724 697 ± .36843 448i	1.31966 923 ∓ 1.04058 814i
	5	.53392 634 + .00000 000i	1.87291 752 + .00000 000i
6	1, 2	.08786 626 ± .92009 404i	.10285 254 ∓ 1.07702 331i
	3, 4	.39416 727 ± .46819 799i	1.05229 916 ∓ 1.24993 725i
	5, 6	.49826 825 ± .14769 920i	1.84484 830 ∓ .54685 928i
7	1, 2	−.00076 496 ± .96470 825i	−.00082 196 ∓ 1.03658 217i
	3, 4	.32727 973 ± .54346 944i	.81317 581 ∓ 1.35033 175i
	5, 6	.45588 935 ± .25464 118i	1.67190 107 ∓ .93385 569i
	7	.49224 949 + .00000 000i	2.03149 016 + .00000 000i
8	1, 2	−.07902 919 ± 1.00066 480i	−.07843 500 ∓ .99314 110i
	3, 4	.26601 917 ± .60293 762i	.61252 402 ∓ 1.38829 762i
	5, 6	.41223 251 ± .33698 985i	1.45409 421 ∓ 1.18868 595i
	7, 8	.47182 912 ± .10911 533i	2.01181 677 ∓ .46525 329i
9	1, 2	−.14919 526 ± 1.03046 752i	−.13761 844 ∓ .95050 835i
	3, 4	.20966 304 ± .65149 353i	.44761 307 ∓ 1.39088 424i
	5, 6	.36931 455 ± .40305 392i	1.23580 348 ∓ 1.34870 244i
	7, 8	.44525 659 ± .19444 915i	1.88616 975 ∓ .82371 403i
	9	.46815 071 + .00000 000i	2.13606 428 + .00000 000i
10	1, 2	−.21284 773 ± 1.05570 953i	−.18351 682 ∓ .91023 036i
	3, 4	.15754 418 ± .69213 469i	.31266 794 ∓ 1.37363 580i
	5, 6	.32790 360 ± .45764 025i	1.03454 187 ∓ 1.44386 337i
	7, 8	.41610 417 ± .26374 950i	1.71443 371 ∓ 1.08670 151i
	9, 10	.45488 509 ± .08636 297i	2.12187 330 ∓ .40285 183i

The refinement of the factors and zeros of $\phi_n(z)$ by Bairstow's method was done by Mrs. Genevieve Mullin Kimbro.

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1. H. E. SALZER, "Orthogonal polynomials arising in the numerical evaluation of inverse Laplace transforms," *MTAC*, v. 9, 1955, p. 164–177.
2. H. E. SALZER, "Equally weighted quadrature formulas over semi-infinite and infinite intervals," *Jn. Math. and Physics*, v. 34, 1955, p. 54–63.
3. W. E. MILNE, *Numerical Calculus*, Princeton University Press, Princeton, New Jersey, 1949, p. 53–57.
4. F. W. J. OLVER, "The evaluation of zeros of high-degree polynomials," Roy. Soc. London, *Phil. Trans.*, v. 244, Ser. A, 1952, p. 385–415.

An Open Formula for the Numerical Integration of First Order Differential Equations

1. Introduction. The integration scheme given below combines the accuracy of multi-point formulas with the convenience, lack of starter formulas, etc., of two-point formulas, as described, for example, by Milne [1]. The price paid is the necessity of evaluating the right hand side of

$$y' = f(x, y) \tag{1}$$

at points which may lie outside the range of integration. It is the writer's belief that this is compensated for by the ease of programming the scheme for a digital computer.

2. The Integration Formulas. For each $n = 3, 4, 5, \cdots$, there exists a matrix D_n of order n and rank $n - 1$ such that if $\bar{y} = (y_0, y_1, \cdots, y_{n-1})$ is a vector of values of the function $y(x)$ at equally spaced points $x_0, x_0 + h, \cdots, x_0 + (n - 1)h$, then $\bar{y}' = (y_0', y_1', \cdots, y'_{n-1})$ is given by

$$\bar{y}' = D_n\bar{y} + \bar{E} \tag{2}$$

where $\bar{E}$ is an error vector whose elements are $O(h^{n-1}y^{(n)}(\xi))$.

Recurrence formulas for the propagation of the solution of (1) can be obtained by setting

$$D_n\bar{y} + \bar{E} = f(\bar{x}, \bar{y}) \tag{3}$$

where

$$f(\bar{x}, \bar{y}) \equiv (f(x_0, y_0), f(x_1, y_1), \cdots, f(x_{n-1}, y_{n-1})),$$

from which all but two variables can be eliminated successively, yielding the desired recurrence relation. Thus for $n = 3$,

$$y_0' = \frac{1}{2h}(-3y_0 + 4y_1 - y_2) + \frac{h^2}{3}y^{(3)} = f(x_0, y_0) \tag{4.1}$$

$$y_1' = \frac{1}{2h}(-y_0 + y_2) - \frac{h^2}{6}y^{(3)} = f(x_1, y_1) \tag{4.2}$$

$$y_2' = \frac{1}{2h}(y_0 - 4y_1 + 3y_2) + \frac{h^2}{3}y^{(3)} = f(x_2, y_2). \tag{4.3}$$

Solving (4.2) for y_2, inserting in (4.1) we get

$$y_1 - y_0 = \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1)\} - \frac{h^3}{12}y^{(3)}(\xi), \tag{5}$$

the familiar trapezoidal rule.

With $n = 4$ we eliminate y_2, y_3 from

$$(6.1) \qquad y_0' = \frac{1}{6h}(-11y_0 + 18y_1 - 9y_2 + 2y_3) - \frac{h^3}{4}y^{(IV)} = f(x_0, y_0)$$

$$(6.2) \qquad y_1' = \frac{1}{6h}(-2y_0 - 3y_1 + 6y_2 - y_3) + \frac{h^3}{12}y^{(IV)} = f(x_1, y_1)$$

$$(6.3) \qquad y_2' = \frac{1}{6h}(y_0 - 6y_1 + 3y_2 + 2y_3) - \frac{h^3}{12}y^{(IV)} = f(x_2, y_2)$$

$$(6.4) \qquad y_3' = \frac{1}{6h}(-2y_0 + 9y_1 - 18y_2 + 11y_3) + \frac{h^3}{4}y^{(IV)} = f(x_3, y_3)$$

getting

$$(7.1) \qquad y_1 - y_0 = \frac{h}{12}[5f(x_0, y_0) + 8f(x_1, y_1) - f(x_2, y_2)] + \frac{h^4}{24}y^{(IV)}$$

$$(7.2) \qquad y_2 = 5y_0 - 4y_1 + 2h\{f(x_0, y_0) + 2f(x_1, y_1)\} + \frac{h^4}{6}y^{(IV)}.$$

Replacing y_2 in (7.1) by (7.2) and using the mean value theorem we get finally

$$(8.1) \qquad y_1 - y_0 = \frac{h}{12}\{5f(x_0, y_0) + 8f(x_1, y_1) - f(x_2, y_2^*)\}$$
$$O\frac{h^4}{24}y^{(IV)}(\xi)\left[1 + \frac{h}{3}\frac{\partial f}{\partial y}(x_2, \eta)\right]$$

where

$$(8.2) \qquad y_2^* = 5y_0 - 4y_1 + 2h[f(x_0, y_0) + 2f(x_1, y_1)]$$

$$(8.3) \qquad x_0 \leq \xi \leq x_2$$

and η lies between y_2 and y_2^*.

If $f(x, y)$ is linear in y then y is given explicitly by (8.1) and (8.2); otherwise an initial guess may be made of y_1, y_2^* computed from (8.2), and a new y_1, computed from (8.1), repeating the process until convergence is reached before proceeding to the next point.

In the linear case, (1) takes the form

$$(9) \qquad y'(x) = P(x) + Q(x)y(x),$$

and (8.1), (8.2) become

$$(10) \qquad y_1\left[1 - \frac{h}{3}(2Q_1 + Q_2) + \frac{h^2}{3}Q_1Q_2\right] - y_0\left[1 + \frac{h}{12}(Q_0 - Q_2) - \frac{h^2}{6}Q_0Q_2\right]$$
$$= \frac{h}{12}[5P_0 + 8P_1 - P_2] - \frac{h^2Q_2}{6}[P_0 + 2P_1] + \frac{h^4}{24}y^{(IV)}(\xi)\left[1 + \frac{h}{3}Q_2\right].$$

The method clearly generalizes to $n > 4$ yielding formulas of higher order accuracy. The involved calculations necessary in carrying out the propagation, however, may make the method too inconvenient for machine calculation.

3. An example. We illustrate the method with the solution of

$$\begin{cases} y' = 1 + y \\ y(0) = 2 \end{cases} \tag{11}$$

for which the exact solution is $y = 3e^x - 1$.

The functions $P(x)$, $Q(x)$ are here identically unity, and (10) becomes

$$\left[1 - h + \frac{h^2}{3}\right] y_1 = \left[1 - \frac{h^2}{6}\right] y_0 + h - \frac{h^2}{2} \tag{12}$$

or

$$y_1 = A(1 + y_0) - 1 \tag{13}$$

where

$$A = [1 - \tfrac{1}{6}h^2][1 - h + \tfrac{1}{3}h^2]^{-1}.$$

The fourth order Runge-Kutta method also gives (13) with

$$A = [1 + h + \tfrac{1}{2}h^2 + \tfrac{1}{6}h^3].$$

For $h = .05$ the solution was propagated twenty steps with the following results:

x	y (Runge-Kutta)	y (Eq. 10)	y (Exact)
0.00	2.00000 0000	2.00000 0000	2.00000 0000
0.20	2.66420 4603	2.66420 4351	2.66420 8275
0.40	3.47546 5123	3.47546 4508	3.47547 4093
0.60	4.46633 9967	4.46633 8842	4.46635 6401
0.80	5.67659 6021	5.67659 4190	5.67662 2786
1.00	7.15480 4623	7.15480 1828	7.15484 5486

4. Significance. In order to obtain accuracy of higher order than the third, using the Runge-Kutta technique, it is necessary to evaluate the functions $Q(x)$, $P(x)$ in equation (9) at points interior to each mesh interval. On the other hand, equation (10) requires that evaluation only at points at which y is to be calculated, plus one extra beyond the right hand limit of integration.

This fact, plus the freedom from starter formulas required by some multi-point methods combine to enhance the value of equation (10), or equivalently (8.1), (8.2) for machine computation.

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1. W. E. MILNE, *Numerical Calculus. Approximations, Interpolation, Finite Differences, Numerical Integration, and Curve Fitting*, Princeton Univ. Press, New Jersey, 1949, p. 96–98.

Note on "A Method for Computing Certain Inverse Functions"

The method for computing certain inverse functions, one binary digit at a time, which was described by D. R. Morrison in a recent issue of *MTAC* [1] has been used in this laboratory. In particular, a subroutine for computing an inverse cosine, $x = \arccos y$, based on the method was given in [2] (part III, subroutine T4, p. 152–153). It was, however, pointed out by van Wijngaarden [3] that the method gives poor accuracy for certain values of the argument, namely, those for which one or more of the functions $\cos x$, $\cos 2x$, $\cos 4x \cdots \cos 2^k x$ are near unity. When x is near zero the error is, perhaps, of little importance since the equation $x = \arccos y$ does not then determine x with any great precision, but this is not the case when x is near $\pi/2$, $\pi/4$, etc. In general abnormally large errors may occur if, in Morrison's notation,

$$dy_n/dx = 0 \text{ for any } n \leq N,$$

since δ will then be of order $\sqrt{\epsilon}$ if $d^2y_n/dx^2 \neq 0$, and of larger order otherwise. The number of correct figures in the value obtained for an inverse cosine or similar function may, as a result, be only about half as many as Morrison suggests.

Although, in cases in which the above objection does not apply, digit by digit methods of computing functions may sometimes be useful in a digital computer on account of their simplicity, they are, in general, slow in operation, and unless storage capacity is very restricted other methods are, in our experience, generally to be preferred.

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1. D. R. Morrison, "A method for computing certain inverse functions," *MTAC*, v. 10, 1956, p. 202–208.
2. M. V. Wilkes, D. J. Wheeler, & S. Gill, *The Preparation of Programs for an Electronic Digital Computer*, Addison-Wesley Press, Cambridge, Mass., 1951.
3. A. van Wijngaarden, "Erreurs D'Arrondiment dans les Calculs Systématiques," Centre National de la Recherche Scientifique, *Colloques Internationaux*, v. 37, 1953, p. 285–293.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

75[A].—Walter Schmidt, *Der Rechner*, Technischer Verlag Herbert Cram, Berlin, 1955, xix + 200 p. DM 18.00.

This gives on page n, $n = 1(1)200$, the product mn, to one place of decimals, where $m = p + \theta$, $p = 0(1)100$, $\theta = 0.1(.1).9$; $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{1}{4}$, $\frac{3}{4}$. The format is satisfactory and the printing tolerable. The accuracy has not been checked.

There is a rather lighthearted introduction, which gives various examples of the use of the table.

J. T.

This review was prepared by J. Todd for *Mathematical Reviews*.

76[A].—Franz Triebel, *Rechen-Resultate*. Achte Auflage, Technischer Verlag Herbert Cram, Berlin, 1956, ii + 285 p. DM 26.00.

This gives, on page $n + 2$ for $n = 1(1)100$, the product mn to two decimal places, where $m = 1(1/4)100$. There follows the product mn for $m = 101(1)1000$, $n = 1(1)100$, each page covering five values of m; there is also a table of mn, $m = 1(1)300$, $n = \frac{1}{6}, \frac{1}{3}, \frac{2}{3}$.

There is a brief introduction showing the use of the tables. There are elaborate thumb indices. Printing is satisfactory; the accuracy has not been checked.

J. T.

This review was prepared by J. Todd for *Mathematical Reviews*.

77[B].—H. Nagler, *Table of Square Roots of Integers*, on microfilm, 101 frames, deposited in UMT File.

This table gives $\sqrt{N}$ to 15D (17S) for $N = 1(1)10,000$. The table was computed on an Elliott 402 digital computer, and printed on an automatic printer from five-hole punched paper tape. The author states that the method of computation insures that the error is within one unit in the last decimal place. A random spot check by the reviewer with 18D hand computed values revealed an error of 1 in the last place in two cases out of ten, the other eight cases being correct.

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78[B, C, D, E].—V. Belevitch & F. Storrer, "Le calcul numérique des fonctions élémentaires dans la machine mathématique IRSIA–FNRS," Acad. r. de Belgique, *Cl. d. Sciences, Bull.*, s. 5, v. 52, 1956, p. 543–578.

This article treats in detail the problems of approximating elementary functions by polynomials, for use on a digital calculator using floating decimal arithmetic. The specific calculator the authors have reference to is the Belgian machine ISRIA–FNRS [1], which uses a floating decimal arithmetic with 15 significant digits in the mantissa, and an exponent between -50 and 50. The basic command structure allows the machine to add, subtract, and multiply; all other arithmetic operations must be programmed.

It is clear that in dealing with floating point operations, the desirable criterion for an approximating polynomial is that the *relative* error, rather than the *absolute* error, of the approximation be bounded. Two methods of deriving a polynomial with a given relative error from a polynomial with a given absolute error are discussed; the first method depends on the property that if the function $f(x)/x$ is approximated by a polynomial $p(x)$ on an interval I with an absolute error $= e$, then the approximation $xp(x)$ to the function $f(x)$ will have a *relative* error $= Me$, where $m = \sup|x/f(x)|$, for x in I. The second method utilizes the fact that if the derivative $f'(x)$ is approximated by a polynomial with a certain absolute

error, then the integral of the polynomial will approximate $f(x)$ with a bounded relative error.

Several methods of obtaining the approximating polynomials are discussed, and illustrated by deriving the polynomials used in the calculation of $1/x$, $x^{-\frac{1}{2}}$, $\sin x$, $\arctan x$, 10^x, $10^x - 1$, $\log_{10} x$. Tables are given of the coefficients of the polynomials derived, to 15S.

An analysis of the rounding error in the calculation of polynomials is also made, and the question of which elementary functions should be chosen as basic is investigated. For example, for x near zero, the function $10^x - 1$ is chosen as the basic function, and 10^x is gotten from it by addition of unity; this has obvious advantages in the calculation, for example, of $\sinh x$ for x near zero.

The basic law of floating point coding is "avoid subtracting two nearly equal numbers," and the authors have shown ingenuity in complying with this law while calculating the elementary functions. The ideas contained in this article should prove valuable to everyone who is coding in floating decimal (or floating binary).

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1. OFFICE OF NAVAL RESEARCH, Department of the Navy, *A Survey of Automatic Digital Computers*, Washington, D. C., 1953, p. 53.

79[D].—L. S. KHRENOV, *Pyatiznačnye tablicy trigonometričeskikh funkcii s argumenton, vyrazennym v časovoi mere* (*Five-place tables of the trigonometric functions with argument expressed in hourly measure*). Second edition. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954, 172 p. Price 7.70 rubles.

The main table gives values of the six trigonometrical functions at interval of 4 seconds of time (i.e., 1 minute of arc), usually to 5S, in the range 0 to 3 hours. (Whenever the leading figure is unity, five further digits are given.) First differences are usually given, with their proportional parts alongside the tables. An auxiliary table, new in this second edition, gives $\cot x$, $\operatorname{cosec} x$ to 5S for $x = 0(.1^s)8^m(1^s)40^m$. There is a table of $\sin^2(\frac{1}{2}x)$ for $x = 0(4^s)12^h$. Entries with terminal 5 are marked with $\pm$ to indicate in what direction rounding should be made. There is a collection of conversion tables, constants and formulae from plane and spherical trigonometry and there are worked examples showing the direct and inverse use of the tables. The tables are clearly printed.

There are no references to sources, nor is there a description of the construction. It is stated that the entries are correct to half a unit in the last place. The tables should be convenient for those who require something between the two volumes issued by L. J. Comrie [1] which gives 4+ decimals at 10^s interval, and the British Nautical Almanac Office [2] which gives 7D at 1 interval.

J. T.

This review was prepared by J. Todd for *Mathematical Reviews*.

1. L. M. MILNE-THOMSON & L. J. COMRIE, *Standard Four-Figure Mathematical Tables*, Edition B, MacMillan & Co., Ltd., London, 1931; Edition A, MacMillan & Co., Ltd., London, 1944.
2. H. M. NAUTICAL ALMANAC OFFICE, *Seven-Figure Trigonometrical Tables for Every Second of Time*, H. M. Stationery Office, London, 1939.

80[E, H, P].—Jørgen Rybner, *Nomogrammer over komplekse hyperbolske funktioner* (*Nomograms of Complex Hyperbolic Functions*), Jul. Gjellerups Forlag, Copenhagen, 1955, 39 + 60 p. of illustrations, 30 cm. Price Dan. Kr. 44.00.

This is a second edition of this useful book. (The first edition was reviewed by K. G. Van Wynen, RMT **526**, *MTAC*, v. 3, 1948, p. 174–175.) The unhandy binding of the first edition has been improved, errors have been removed, notation has been changed (complex arguments are now $A + jB$ instead of $b + ja$), nomograms have been added extending the range of the argument for hyperbolic tangent, and nomograms for interaction loss and interaction phase shift have been added.

In addition to discussion and formulas, alignment charts are included for sinh $(A + iB)$ and cosh $(A + iB)$, $0 \le A \le 4$, for tanh $(A + iB)$, $0 \le A \le 3$ and for various incidental functions as follows:

$$x + iy = r(\cos\theta + i\sin\theta);$$

$$R(\cos\alpha + i\sin\alpha) = 1 + r(\cos\theta + i\sin\theta), \quad r \le 1;$$

$$A_r = \ln\left(\frac{1+Z}{2\sqrt{Z}}\right), \quad 1 \le |Z| \le 10;$$

$$B_r = \arg\left(\frac{1+Z}{2\sqrt{Z}}\right);$$

$$A_z = \tfrac{1}{2}\ln(1 - 2e^{-2A}\cos 2B + e^{-4A}), \quad 0 \le A \le 1, \quad 0 \le B \le 90°;$$

$$B_z = \arctan\frac{\sin 2B}{e^{2A} - \cos 2B}, \quad A \ge 0;$$

$$f = \frac{1}{2\pi\sqrt{LC}}, \quad 10^{-9} \le L \le 10^{5} \text{ henry}, 10^{-12} \le C \le 10 \text{ farad}$$

$$K = \sqrt{\frac{L}{C}}, \quad 10^{-9} \le L \le 10^{5}, \quad 10^{-12} \le C \le 10.$$

The tables and their units are chosen for convenience in electrical filter design and other similar problems. Accuracy is adequate for most such applications.

The author notes the following errata:

"On the four new charts for tgh $(A + jB) = r/\theta$ covering the ranges A = 2,00–2,25; 2,25–2,50; 2,50–2,75; 2,75–3,00 nepers, the $B_1 \cdots B_4$ scales are erroneously marked $a_1 \cdots a_4$.

In the list of contents, page 6, the formula 7 for the interaction loss should read:

$$A_z = \tfrac{1}{2}\ln(1 - 2e^{-2A}\cos 2B + e^{-4A}).$$

In the Danish preface, page 9, line 7 from the bottom, the word monogrammerne should read nomogrammerne."

C. B. T.

81[E].—HOMER S. POWLEY, *Table of log cosh x*. One typewritten sheet, 28 cm., deposited in the UMT FILE.

This table lists log cosh x, 3D, for $x = 10(.5)20(5)40$.

Because of the high value of x, the table is essentially $x \log e - \log 2$.

C. B. T.

82[F, Z].—CARL-ERIK FRÖBERG, *Hexadecimal Conversion Tables*, C. W. K. Gleerup, Lund, Sweden, 1957, 26 p., 22 cm. Price 3 kr.

A revised version of the conversion table so widely used around computers with hexadecimal input. (See RMT **1042**, *MTAC*, v. 7, 1953, p. 21.) In this edition the binary point follows the first binary digit of fractions; the first digit is used as a sign. This is clearly important in use of the table; for some machines the numbers must be doubled and the sign inserted properly.

Using A, B, C, D, E, and F for the hexadecimal digits ten through fifteen respectively the table lists:

1. Decimal and hexadecimal integers over the decimal ranges $1(1)1024(16)4096$ and $10^k(10^k)10^{k+1}$, $k = 2(1)12$;
2. Hexadecimal equivalent of decimal fractions x, $x = 10^{-k}(10^{-k})10^{-k+2}$, $k = 2(2)16$;
3. Conversion of $n \cdot 10^k$, to normalized hexadecimal numbers, $n = 1(2)9$, $k = -12(1)12$ and conversion of 10^k and 10^{-k} for $k = 13(1)25$;
4. Hexadecimal form of constants frequently met;
5. Decimal form of hexadecimal fractions $x = 16^{-k}(16^{-k})16^{-k+1}$ (subject to the sign convention mentioned earlier) $k = 1(1)10$.

For similar octal-decimal tables, see van Wijngaarden [1], Karst [2], and Causey [3]. None seems to have been widely distributed.

The following erratum was communicated to the author by S. Arkéus:

On page 23, *for* ${}^2\log 10$ = 6A4D3 C25E8 1209D 82 *read* 6A4D3 C25E6 8DC58 82.

C. B. T.

1. A. VAN WIJNGAARDEN, "Decimal-binary conversion," Report R-130 of the Computation Department, Mathematical Center at Amsterdam (see following review **83**).

2. EDGAR KARST, *Tables for converting 4 digit decimal fractions to periodic octal fractions*. [See Review **6**, *MTAC*, v. 10, 1956, p. 37.]

3. ROBERT L. CAUSEY, *Decimal to octal and octal to decimal conversion tables*. [See Review **65**, *MTAC*, v. 10, 1956, p. 227.]

83[F, Z].—A. VAN WIJNGAARDEN, *Decimal-Binary Conversion and Deconversion*, Report R-130 of the Computation Department of the Mathematical Centre, Amsterdam, 1951, 41 p., mimeographed, 33 cm.

This useful table, hard to read, gives decimal equivalents of octal numbers 0(10)303230 (all digits OCTAL!). In addition it gives decimal values of 2^n, $n = 1(1)50$, exact, decimal values of 2^{-n}, $n = 1(1)50$, 20D, octal values of 10^n, $n = 1(1)18$, exact, and octal values of 10^{-n}, $n = 1(1)18$, to twenty octal digits.

References to similar tables and their usage may be found in Review **82**, above.

C. B. T.

84[F, X].—P. DAVIS & P. RABINOWITZ, "Abscissas and weights for Gaussian Quadratures of High Order," NBS *Jn. of Research*, v. 56, 1956.

This paper contains 20D values of weights and abscissas for Gaussian quadrature rules with $n = 2, 4, 8, 16, 20, 24, 32, 40$, and 48 points. Corresponding values are available from the authors for the cases $n = 64, 80$, and 96.

The weights, a_{kn}, and abscissas, x_{kn}, enter the approximate formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=1}^{n} a_{kn} f(x_{kn}).$$

These numbers, which have been adequately checked by the National Bureau of Standards, should prove useful in cases where: (a) the integrand $f(x)$ can feasibly be computed for arbitrary x and (b) a high degree of accuracy (result is exact if $f(x)$ is a polynomial of degree $\leq 2n - 1$) is needed.

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85[I, X].—K. A. KARPOV, *Tablitsy Koėffitsientov interpoliatsionnoĭ Formuly Lagranzha* (*Tables of Lagrangian Interpolation Coefficients*), Akad. Nauk SSSR, Moscow, 1954, 79 p., 26 cm. Price (including [3]) 61 rubles.

These tables contain for four-point Lagrangian interpolation the coefficients $A_i(t), i = -1(1)2, t = -1(.001)2$ and for five-point interpolation the coefficients $A_i(t), i = -2(1)2, t = -2(.001)2$, 6D. It was issued as a supplement to [3].

Several entries were checked against the Mathematical Tables Project's more extensive tables [1], and no discrepancies were found. The printing of the present volume is clear and easy to read, and the small size of the volume makes the tables much handier than [1] for the many times when four- or five-point interpolation with these increments suffices.

The numbers listed seem to have been rounded individually rather than as a group, so that $\Sigma_i A_i(t)$ may differ from 1 in the last digit. This happens for $t = 0.001$ in the four-point coefficients, for example, where the entries are $A_{-1} = -0.00033\ 3$, $A_0 = 0.99949\ 9$, $A_1 = 0.00100\ 0$, and $A_2 = -0.00016\ 7$. For a slowly varying function tabulated to many places subtraction of a constant (usually common leading digits) may be required to reduce the absolute value of the tabulated entries in order not to introduce a rounding error; restitution of the subtracted portion must follow the interpolation. This process is not always convenient, and the reviewer would prefer the forced rounding used in [1] to assure that $\Sigma_i A_i(t) = 1$. It would seem to be reasonable to mark each digit which should be rounded up in a further truncation of the coefficients so that this unit sum could be maintained; thus if only 3D values are needed the user would round up an overscored third digit and leave others as printed in the table. The additional cost of printing might well be justified in ease of using the tables.

An estimate of the continuing need for Lagrangian interpolation is made in the *MTAC* review of [1]. More extensive tables are available in [1] (and this is

noted in the present work), and tables for sexagesimal arguments are available in NBS AMS No. 35, [2].

C. B. T.

1. NYMTP, A. N. LOWAN, technical director, *Tables of Lagrangian Interpolation Coefficients*, Columbia University Press, New York, 1944. [RMT 162, *MTAC*, v. 1, p. 314–315.]
2. NBS Applied Mathematics Series, No. 35, *Tables of Lagrangian Coefficients for Sexagesimal Interpolation*, U. S. Govt. Printing Office, Washington, D. C., 1954. [Rev. 54, *MTAC*, v. 11, p. 108.]
3. K. A. KARPOV, *Tablitsy Funktsii* $w(s) = e^{-s^2} \int_0^s e^{x^2} dx$ *v Kompleksnoi oblasti*, Akad. Nauk SSSR, Moscow, 1954.

86[K].—BELL AIRCRAFT CORPORATION, *Table of Circular Normal Probabilities*, Operations Analysis Group, Dynamics Section, Report No. 02-949-106, 1956, iv + 305 p., 22 × 27 cm. (oblong). A limited number of copies are available by writing to the Research Division, Bell Aircraft Corp., Buffalo 5, New York. One copy deposited in the UMT FILE.

Circular normal probability integrals

$$P(T, \sigma) = \frac{1}{2\pi\sigma^2} \int\int_{x^2+y^2 \leq R^2} e^{-\frac{(x-a)^2+(y-b)^2}{2\sigma^2}} dydx$$

for $R/\sigma = 0(.01)4.59$ and $\sqrt{a^2 + b^2}/\sigma = D/\sigma = 0(.01)3$, 5D.

These tables were calculated on an IBM model 650 computer and checked against other available tables with perfect agreement over regions of overlap. Two thousand values chosen randomly were checked and first and second differences were examined.

The calculation is described and a most elementary illustrative example given. Printing by a photographic offset process is adequate.

No differences or aids to interpolation are given.

C. B. T.

87[L].—T. PEARCEY, *Table of the Fresnel Integral to Six Decimal Places*, Cambridge, England, at the University Press, 1957, 63 p., 24 cm. Price $2.50.

This table was compiled for, and printed by, the Commonwealth Scientific and Industrial Research Organization, which published an Australian edition at Melbourne in 1956. The functions tabulated are

$$C = \frac{1}{2}\int_0^x J_{-\frac{1}{2}}(t)dt = \frac{1}{\sqrt{(2\pi)}}\int_0^x \frac{\cos t}{\sqrt{t}} dt$$

$$S = \frac{1}{2}\int_0^x J_{\frac{1}{2}}(t)dt = \frac{1}{\sqrt{(2\pi)}}\int_0^x \frac{\sin t}{\sqrt{t}} dt.$$

Tables with argument u, where $x = \frac{1}{2}\pi u^2$, are also common, but the argument of the present table is x, though a few expansions in terms of u are displayed in the explanatory text. The values of C and S are listed to 7D for $x = 0(.01)1$ and to 6D for $x = 1(.01)50$, with δ^2 throughout.

The values were mostly computed by subtabulation of a 7D table for

$x = 0(.02)1$ given by Watson [1], and a 6D table for $x = 1(.5)50$ due to Lommel, and reproduced by Watson; Lommel's table is unreliable in the sixth decimal, but was differenced and corrected by recomputation where necessary. The compiler states that "errors in rare cases may amount to 1 unit in the last figure."

The reviewer compared the values up to $x = 2$ with 8D values (with possible error up to 2 final units) computed by Corrington [2]. Apart from a number of cases (almost all in the 7D portion) in which the two tables differ by between 0.5 and 0.9 units of the last place printed in the Australian tables, the comparison revealed the following errors, all in S:

x	*For*	*Read*
0.05	.00298 12	.00297 30
0.07	.00492 43	.00492 40
1.97	.55507 4	.55507 3

These three corrections were verified by the reviewer by independent computation; the error in the third case is about 1.3 final units. The compiler states that some values for small arguments were specially computed; it appears that this part of the work was not well done. On the other hand, it is surprising that a 6D table produced by subtabulating a 6D table should contain so *few* rounding errors in the portion tested; perhaps the compiler knows more about the seventh decimal than he has claimed. The second differences appear to be modified when necessary, though this fact is not stated. The table does not pretend to be definitive, but will nevertheless be found very useful.

A. F.

1. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press, England, 1944.

2. M. S. CORRINGTON, *Tables of Fresnel Integrals, Modified Fresnel Integrals, the Probability Integral, and Dawson's Integral*, Radio Corporation of America, R.C.A. Victor Division. [*MTAC*, v. 7, UMT 166, 1953, p. 189.]

88[L].—M. FERENTZ & C. HARRISON, "A tabulation of the function $\frac{1}{x}\int_0^x J_0(y)dy$, $x = 0(.01)31$; 4D," Argonne National Laboratory, Lemont, Illinois, 8 ozalid sheets, 28 cm. Three copies deposited in the UMT FILE.

This is a table of the function $f(x)$ given in the title for $x = 0(.01)31$, to 4D. The table was computed from Taylor's expansion. Values of $\frac{x}{2}f(x)$ are given for $x = 0(.02)1$ to 7D by G. N. Watson [1], p. 752. For $x = 0(.02)16$, 7D values of $f(x)$ can be computed from tables given by Watson, l.c., p. 666 et seq., using the formula

$$f(x) = J_0(x) + \frac{\pi}{2}\,(J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)).$$

P. HENRICI

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1. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.

89[P].—R. E. D. Bishop & D. C. Johnson, *Vibration Analysis Tables*, Cambridge University Press, New York, 1956, viii + 59 p., 28 cm. Price $2.00.

In writing their book, *Mechanics of Vibration*, Cambridge University Press, the authors collected works from various previous engineering publications and managed, in part, the computation of formulae and numerical tables intended to help the analysis (and synthesis) of conservative mechanical systems. These aids, isolated from the main volume, have an independent value for a professional analyst; accordingly they have been extracted in this booklet form.

The structural elements, of which the more complicated systems are assumed to be composed, and their motions are: lateral vibration of a taut, uniform string; torsional vibration of a uniform circular shaft; longitudinal vibration of a uniform bar; flexural vibration of a uniform beam. For the three first cases, where the motion is governed by the same second order differential equation, the receptances, natural frequencies, and modes of vibration are given in terms of structural dimensions for three combinations of the simplest boundary conditions, namely, either the distortion or its longitudinal derivative vanishing at the ends. In giving the receptance formulae for the flexural vibration, the boundary conditions considered are those of a beam with clamped, pinned, sliding, or free ends in all proper combinations. The related functions, composed of products and product sums of trigonometric and hyperbolic functions are given to 5S for $x = 0(.05)11$. For the boundary condition combinations, pinned-pinned, clamped-clamped, free-free, clamped-free, clamped-pinned, and free-pinned the characteristic functions and their derivatives up to the fifth mode are given to 5S for $x/l = 0(0.02)1$, and likewise the five first roots of the associated characteristic equations.

These lucid tables form a valuable contribution to the similar and previous German and Russian collections.

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90[U].—Einar Anderson (Director), Torben Krarup, & Bjarner Svejgaard, *Geodetic Tables, International Ellipsoid*, Geodaet. Inst. Skr. (Mémoires de l'Institut Géodésique de Danemark), Ser. 3, v. 24, Copenhagen, 1956, 8 p. introduction + 184 p. tables, 28.5 cm.

Geodesists employ tables based on a standard assumed ellipsoid of revolution to extrapolate latitudes and longitudes from an astronomically determined point to new points. This procedure was originally worked out in the eighteenth century as a method of determining the figure of the earth by comparing a series of astronomical determinations with the results to be expected from some assumed shape of the earth. It was extended to the general mapping of the European countries as the original arcs of the meridian were expanded into the modern national triangulation nets.

The present tables are for the most part carried to a precision of 12 decimal places or 12 significant figures. Since the question of the requirements for such precision have in the past been discussed in *Mathematical Tables and Other Aids to Computation*, it may not be out of place to discuss the problem here.

The results of precise triangulation have a precision of the order of 0".3 in the

angles, or six-figure accuracy. Since this accuracy may be demanded between points only a few kilometers apart, it may, under some circumstances, imply accuracies of a few millimeters. Since 1 millimeter is $0\overset{''}{.}00004$ of latitude, the use of five decimals of seconds of arc may, under some circumstances, be required.

At the same time, the triangulations of rather large areas, such as Europe or North America, have been brought to consistency by the simultaneous adjustment of the measured angles in very large systems of equations. Hence the latitudes and longitudes within the scheme must be treated formally as though they represented measurements with a precision of 10 or 11 significant figures.

The problem comes to a head when it is necessary to transform the latitudes and longitudes into a coordinate system of the type which is used for military or cadastral surveys. There are only two ways to avoid the use of a large number of significant figures: first, to permit the precision of the computations to suffer, by rounding off. This loses the power of the coordinates to define the direction and distance between nearby points. Second, to make use of small systems, thereby risking trouble at the junctions. Neither is desirable; and hence the requirement for precise tables.

It might be thought that the tables, although formally precise, need not be theoretically correct. An outstanding example of this scheme of thought is the tables for the Lambert Nord de Guerre, prepared in 1916 for France. Of the five constants at the head of this table, no three could be made to agree; and no two agreed with the table itself. This was not serious until, in 1943, it became necessary to extend the tables. One Allied agency extended them by differencing the tables and extrapolating; another solved from the tables for the constants of the spheroid which would produce the given tables; while a third managed to guess the approximation which the originators of the table had made. Any one of these tables would have done the job, but the discrepancies between them meant that they could not be used together. Ultimately one was adopted.

The present, very precise tables are based on the International Ellipsoid, adopted at Madrid in 1924, with the dimensions:

a (equatorial radius)	6,378,388 meters
f (flattening)	1/297.

They replace logarithmic tables prepared at that time. They include:

$W = \sqrt{1 - e^2 \sin^2 \phi}$, $\phi = 0°(1')90°$, 12D,
$10^7/N = 10^7 W/a$, in meters^{-1}; $\phi = 0°(1')90°$, 12S,
$10^7/M = 10^7 w^3/a(1 - e^2)$ in meters^{-1}; $\phi = 0°(1')90°$, 12S,
$\gamma = g_e \left[W + \dfrac{\omega^2 a C \sin^2 \phi}{g_e W} \right]$ in milligals, $\phi = 0°(1')90°$, 8S,
$\phi - \phi^*$, defined by $\tan(\phi^*/2 + \pi/4) = \tan(\phi/2 + \pi/4) \left(\dfrac{1 - e \sin \phi}{1 + e \sin \phi} \right)^{e/2}$
and expressed in seconds of arc; $\phi = 0°(1')90°$, 6D.
$\beta - \phi$, defined by $\tan \beta = b/a \tan \phi$; $b = a\sqrt{1 - e^2}$
and expressed in seconds of arc; $\phi = 0°(1')90°$, 6D.
$\rho''/2MN$, where $\rho'' = 648,000/\pi$, $\phi = 0°(1')90°$, 6S.

The quantities M and N are the radii of curvature of the ellipsoid along the meridian and parallel to it. They are formed from the auxiliary quantity W. The terrestrial gravity γ, is here defined by the International gravity formula, slightly modified to make it rigorously consistent with the International ellipsoid. Note that the geodesists include centrifugal force in gravity, which they distinguish from gravitation, which is taken as not including centrifugal force.

The Gaussian latitude, ϕ^*, is identical with the quantity called the isometric latitude in the tables of the U. S. Coast and Geodetic Survey and the U. S. Lake Survey. The term isometric latitude is used by the Europeans for $gd^{-1}\phi^*$, the antigudermannian of the Gaussian latitude. The antigudermannian of ϕ^* is proportional to the north distance corresponding to the given latitude on a Mercator map. For a sphere, it reduces to the log tangent of the semi-colatitude. The Gaussian latitude is fundamental for the calculation of conformal projections (especially for coordinate computations).

The reduced latitude, β, is also called the parametric latitude. The meridional arc can be expressed as an elliptic integral of the second kind in terms of the parametric latitude.

The tables have been computed from Fourier series by an IBM 602A, and printed directly from an IBM 416 tabulation. The figures are clear and readable.

A cursory check of the table of $\phi - \phi^*$ against the tables of latitude functions for the International Ellipsoid from the Lake Survey [1] indicated agreement to the 4th place of decimals, the limit of the Lake Survey tables.

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1. War Department, Corps of Engineers, U. S. Lake Survey, *Latitude Transformation, Hayford Spheroid, Geodetic Latitude to Isometric Latitude and Isometric Latitude to Geodetic Latitude*, New York Office, Military Grid Unit, 1944.

91[V].—Speer Products Company, The SPEER BALLISTICS CALCULATOR, Speer Products Company, Lewiston, Idaho, 23 × 10 cm. Price $1.00.

This little instrument is a slide rule for the computation of drop and remaining velocity of small arms bullets in terms of muzzle velocity, range, and ballistic coefficient. The scales were prepared on the basis of Ingall's tables, which were in turn based upon the Siacci approximation to flat trajectories and the Gâvre drag function. As an example, the range scale (versus drop) goes from 50 to 1000 yards, and may be read within about ten percent. Ballistic coefficients are supplied for each of the manufacturer's bullets.

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92[V].—Homer S. Powley, *Extension of Ingalls' Table IIA*. One typewritten sheet, 28 cm., deposited in the UMT File.

For $Z = 20000(1000)45000$ and 50000, the functions $A\,V^2/700^2$, $\log V/u$ and N are listed to 2D, 4D, and 2D respectively. For values of Z divisible by 5000 $A\,V^2/700^2$ is listed to 4D and N to 3D.

Ingall's Tables are described by Bliss [1]. According to the author the present tables are intended to extend the tables for solution of problems relating to some low power weapons.

C. B. T.

1. G. A. Bliss, *Mathematics for Exterior Ballistics*, John Wiley and Sons, New York, 1944, p. 33–35.

93[W, X].—S. Vajda, *The Theory of Games and Linear Programming*, John Wiley & Sons, Inc., New York, 1956, 106 p., 17 cm. Price $1.75.

This handy pocket-sized book contains an exposition of some of the most useful aspects of the theory of two-person zero-sum games and of linear programming.

Any work of this size must necessarily be incomplete. Thus the present volume contains no reports of experience with extensive computations involved with games and linear programming problems. There is no indication of the handling of non-linear situations, and no real full use of the continuous space in which the problems are invented. A final chapter devoted to Beale's "Method of Leading Variables," and a few introductory and incomplete historical remarks might profitably have been omitted in favor of other topics (the genesis of problems, for example), but on the whole there can be no real objections to the choice of material expounded.

A major effort is devoted to a simplex method, which is certainly the most widely used method resolving games and linear programming problems. The description of the theory covered is elementary and clear. However there are no exhibited statements, such as theorems, to summarize the arguments that come to mind.

Several examples are given and solved. The number of drawings included to clarify the presentation is impressive, particularly in the early chapters which are devoted to graphical representations of the theory. This all contributes to the exposition.

In all the book is a valuable contribution to the literature in this popular field, and it contains a valuable selection of material presented clearly.

C. B. T.

94[P, W, X, Z].—*Proceedings of the Second Annual Computer Applications Symposium*, held October 24–25, 1955, sponsored by the Armour Research Foundation of the Illinois Institute of Technology, Chicago, Illinois, 1956, 108 p., 23 cm. Price $3.00.

This is a collection of reports of talks delivered at the symposium, together with the discussion which followed each paper. For all practical purposes, the word "digital" could be in the title of the symposium.

The program was devoted to seven papers having to do with "Computers for Business and Management," and seven concerned with "Computers for Engineering and Research." Twelve papers and one abstract appear in the *Proceedings*, as follows:

Computers for Business and Management: "The use of digital computers in industry," by R. F. Clippinger, "A dollar and cents approach to electronics," by

John L. Marley, "An application of computers to general bookkeeping," by W. F. Otterstrom, "User experiences and applications of the ERA 1103," by George E. Clark, "Automobile selective underwriting and automatic rating on the IBM 650," by C. A. Marquardt, "Cutting costs with linear programming," by Jacob E. Bearman, "Probability forecasts in management decisions (abstract)," by Stanley Reiter.

Computers for Engineering and Research: "Use of the IBM 650 in scientific computations," by A. W. Wymore, "Engineering applications of large scale computers," by C. B. Ludwig, "High speed computation of engine performance," by J. T. Horner, "Pyrolysis reactor design computations," by H. C. Schutt and R. H. Snow, "Aircraft flight test data processing," by T. M. Bellan, "Programming a Monte Carlo problem," by J. F. Hall and J. M. Cook.

The *Proceedings* contain some information which is very useful for those interested in the two broad fields covered.

T. H. Southard

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95[G, H, X].—Richard B. Smith, *Table of Inverses of Two Ill-Conditioned Matrices*, Westinghouse Electric Corporation, Bettis Atomic Power Division, Pittsburgh, Pennsylvania, 1957, ii + 68 p., 29 cm. Deposited in the UMT File.

The listings are the inverses ($n = 2, 3, \cdots, 15$) for $A_n = (a_{ij})$, $a_{1j} = 1$ for $j = 1, 2, \cdots, n$, $a_{ij} = 1/(i + j - 1)$ for $i = 2, 3, \cdots, n$ and $j = 1, 2, \cdots, n$; and $B_n = (b_{ij})$, $b_{ij} = 1/(p + i + j - 1)$ for $p = 0$ and $i, j = 1, 2, \cdots, n$. The matrix A_n is described by M. Lotkin [1] and B_n is described by Savage and Lukacs [2] and Collar [3, 4].

The calculations were carried out on an IBM 650 using a complete double precision rational number interpretive system. The listings are believed to be correct.

Richard B. Smith

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1. Mark Lotkin, "A set of test matrices," *MTAC*, v. 9, 1955, p. 153–161.
2. I. R. Savage & E. Lukacs, "Tables of inverses of finite segments of the Hilbert matrix," NBS Applied Mathematics Series, No. 39, *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues*, 1954, p. 105–108.
3. A. R. Collar, "On the reciprocation of certain matrices," Royal Soc. Edinburgh, *Proc.*, v. 59, 1939, p. 195–206.
4. A. R. Collar, "On the reciprocal of a segment of a generalized Hilbert matrix," Cambridge Phil. Soc., *Proc.*, v. 47, 1951, p. 11–17.

96[H, X].—Zygmunt Dowgird, *Krakowiany i ich zastosowanie w mechanice budowli.* (*Cracovians and their application in structural mechanics.*) Państwowe Wydawnictwo Naukowe, Warszawa, 1956, 168 p. zł. 18.

Cracovians are rectangular arrays of numbers which are added like matrices, but which are multiplied column-by-column. They were developed by T. Banachiewicz, apparently because column-by-column multiplication is easier in desk computing than row-by-column multiplication. The present work expounds the definitions, notations, and properties of cracovian theory, and their use in

problems of linear algebra. The second chapter is devoted to the solution of systems of linear equations by various methods of triangular decomposition. In the third chapter are discussed simple iterative methods for solving linear systems and for computing eigenvalues. The exposition is elementary. There are problems from structural mechanics, and many numerical examples of orders up to 5 or 6. The examples are oriented towards desk computation.

The fact that cracovian multiplication is non-associative causes various strained notations, and appears to the reviewer as an overwhelming impediment to fundamental progress. Nevertheless, cracovians have a minority of enthusiastic supporters, largely but not exclusively in Poland.

G. E. Forsythe

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(After September, 1957, will be at Stanford University, Stanford, California.)
This review was prepared by G. E. Forsythe for *Mathematical Reviews*.

97[S, X].—Sir Harold Jeffreys & Bertha Swirles (Lady Jeffreys), *Methods of Mathematical Physics*, Cambridge University Press, Great Britain, 1956, iv + 714 p., 25 cm. Price $15.00.

This is the third improved edition of this impressive textbook of mathematics as it should be applied to physics. The authors are firm in their feeling that the physicist needs a rigorous proof of all theorems used, and they present such proofs under conditions which are suitable for the physics applications they have in mind.

At the beginning of the chapter on numerical methods the authors quote Lord Kelvin, "I have no satisfaction in formulas unless I feel their numerical magnitude." For purposes of this book the authors have no interest in mathematical theorems unless they are applicable to physics (their stated standard is applicability in at least two branches), and they have no satisfaction unless the theorem is proved under adequate conditions for their application and an application illustrated.

Numerous problems are included.

Much of the work concerns numerical analysis; this varies from the classical studies of finite differences through various uses of analysis and studies of special functions. Thus the book is a valuable but not a complete textbook on many aspects of numerical analysis.

The authors have continued to improve the book through these three editions evidently seeking advice wherever it is available. Thus they have put together a sound book containing much material not easily available elsewhere.

The chapter headings follow:

Chapter 1. The Real Variable, 2. Scalars and Vectors, 3. Tensors, 4. Matrices, 5. Multiple Integrals, 6. Potential Theory, 7. Operational Methods, 8. Physical Applications of the Operational Method, 9. Numerical Methods, 10. Calculus of Variations, 11. Functions of a Complex Variable, 12. Contour Integration and Bromwich's Integral, 13. Conformal Representation, 14. Fourier's Theorem, 15. The Factorial and Related Functions, 16. Solution of Linear Differential Equations of the Second Order, 17. Asymptotic Expansions, 18. The Equations of Potential, Waves, and Heat Conduction, 19. Waves in One Dimension and Waves

with Spherical Symmetry, 20. Conduction of Heat in One and Three Dimensions, 21. Bessel Functions, 22. Applications of Bessel Functions, 23. The Confluent Hypergeometric Function, 24. Legendre Functions and Associated Functions, 25. Elliptic Functions.

There is a reasonably detailed index.

C. B. T.

98[S, X].—F. B. HILDEBRAND, *Advanced Calculus for Engineers*, Prentice-Hall, Inc., New Jersey, 1956, xiii + 594 p., 22 cm. Price $7.75.

The reviewer belongs to a school of thought which holds that the simplest numerical methods (Euler's method of forward differences in ordinary differential equations, for example) may be the soundest introduction to many mathematical subjects. He would have liked to find more numerical material in this text by this author of one of our better numerical analysis texts.

The volume does contain the following tables: A table of 41 Laplace transforms, a table of $\sqrt{\frac{\pi}{2}}\, x^m J_m(x)$ and $\sqrt{\frac{\pi}{2}}\, x^m I_m(x)$ for $m = -\frac{1}{2}(1)9/2$, $\Gamma(x)$, $x = 1(.01)1.99$, 4D, and the first five zeros of $J_p(x)$ $p = 0(1)5$.

A chapter on numerical methods for solving ordinary differential equations includes Taylor's series, the Adams method, the Runge-Kutta method, and the Picard method. A chapter devoted to series solutions of differential equations introduces Bessel functions, Legendre functions, and the hypergeometric function (and the gamma is introduced in connection with Laplace transforms).

Various iterative methods for computing eigenvalues or solutions of equations are discussed, and the author does pay attention to the requirement for numerical answers which engineers frequently face. However, when treating partial differential equations the author does not take up difference equation methods of approximating solutions, which were not popular at the time of writing.

Several topics usually included in advanced calculus texts are omitted; multiple integrals and surface integrals are treated only as they arise in vector analysis.

Problems are carefully chosen.

The book more or less parallels material in the much more ambitious and difficult work on *Methods of Mathematical Physics*, by Jeffreys and Jeffreys [1]. Chapter headings are listed below.

Solutions of Linear Ordinary Differential Equations
The Laplace Transformation
Numerical Methods for Solving Ordinary Differential Equations
Series Solutions of Differential Equations
Boundary-Value Problems and Orthogonal Functions
Vector Analysis
Partial Differential Equations
Solutions of Partial Differential Equations of Mathematical Physics
Functions of a Complex Variable.

C. B. T.

1. HAROLD JEFFREYS & BERTHA SWIRLES JEFFREYS, *Methods of Mathematical Physics*, Cambridge, England, at the University Press, 1946.

99[X, P].—S. H. Crandall, *Engineering Analysis, A Survey of Numerical Procedures*, McGraw-Hill Book Co., Inc., New York, 1956, x + 417 p., 24 cm. Price $9.50.

This textbook has grown out of the author's continuing efforts to acquaint engineering students with numerical methods of attacking problems in engineering analysis. The author describes engineering analysis as the performance of two steps: "1. Construction of a mathematical model for a physical situation. 2. Reduction of the mathematical problem to a numerical procedure."

The present book is largely devoted to the second of these steps, but the arrangement of material conforms more nearly with the first step than with the arrangement of mathematics courses in most schools. Actually, to a mathematician, the book is more nearly a most valuable catalogue of methods (with careful references to existing literature) than a textbook, for proofs are presented only in summary form.

Three classes of problems are considered: Equilibrium problems, Eigenvalue problems, and Propagation problems. Each of these classes is considered first as a lumped-parameter problem (or a problem with a finite number of variables) and then (three chapters later) as a continuous problem.

The reviewer has long felt that use of at least the most elementary numerical methods is the soundest introduction to many courses in mathematics. Euler's method of forward differences for numerical solution of an ordinary differential equation, for example, offers the student a feeling for the properties described in the Picard existence and uniqueness theorem; calculation of difference quotients (particularly when they are stated in terms of displacement and time) seems to be a sound introduction to the derivative, and so on. Thus one important purpose which this book can serve is presenting this feeling concerning the nature of solutions of the problems treated. Just as a student who knows Euler's method can do something with any ordinary differential equation initial value problem he is likely to meet, so the student of the present text may make efficient progress on any problem he is likely to encounter from the fields studied. The study (or at least the perusal) of a book of this type would seem to provide an excellent introduction to many aspects of abstract analysis through the concrete examples furnished. Here the motivation for the non-numerical studies would be the promise that some of the problems can be solved more generally and with less effort—a situation to which many students seem highly attracted.

In any event, since recent engineering problems have more and more demanded numerical solution, the book (or its equivalent, which seems not to exist) seems necessary for any complete training in several fields of engineering, and the reviewer suggests that teachers of more abstract courses might well extract much material from the book for introductory use.

The author has chosen methods which he feels are most likely to be useful in engineering analysis, and he has expounded them carefully and in a scholarly way consistent with the level of maturity at which he aims. In approximate methods he mentions stability considerations carefully, but he does not swamp the student with all technicalities which can crop up in stability studies. In connection with relaxation he mentions overrelaxation, but stops short of the detailed studies available on the subject.

The material to be covered requires a study of methods of solution of systems of linear algebraic equations, and ordinary and partial differential equations with various kinds of initial and boundary conditions. Numerical methods (of scope indicated below) are presented in detail along with an outline of the basic mathematics involved.

The material in the book is illustrated with many worked examples and with a spectacular number of figures and much tabulated material. Problems for the student in each section vary from routine application to problems demanding some knowledge of the theory—determination of degree of convergence of discrete methods, for example.

There is an impressive number of references to basic material used in the book and to additional material which is available for more extensive study.

The book is the second which we have seen recently stressing the importance of numerical calculation in engineering analysis; Purday's book [1] covers much of the same material but with considerably less detail.

A reasonable idea of the contents is gained from the section headings, which follow.

1. *Equilibrium problems in systems with a finite number of degrees of freedom:* Particular examples—Formulation of the general problem—Mathematical properties—Extremum problems—Elimination method for linear systems—Iteration—Relaxation—Iteration combined with elimination—Procedures applicable directly to physical systems.

2. *Eigenvalue problems for systems with a finite number of degrees of freedom:* Particular examples—Matrix notation—Formulation of the general problem—Mathematical properties—An extremum principle for eigenvalues—Direct methods of solution—Iteration—Intermediate eigenvalues—n-step iteration—Relaxation methods—Upper and lower bounds for eigenvalues—Diagonalization of matrices by successive rotations.

3. *Propagation problems in systems with a finite number of degrees of freedom:* Particular examples—Formulation of the general problem—Mathematical properties—Iteration—Series methods—Trial solutions with undetermined parameters—Finite-increment techniques—Introduction to step-by-step integration procedures—Recurrence formulas with higher-order truncation error—Step-by-step integration methods for systems with several degrees of freedom.

4. *Equilibrium problems in continuous systems:* Particular examples—Formulation of the general problem—Mathematical properties—Extremum problems—Trial solutions with undetermined parameters—Finite-difference methods.

5. *Eigenvalue problems in continuous systems:* Particular examples—Formulation of the general problem—Mathematical properties—Extremum principles for eigenvalues—Iteration—Trial solutions with undetermined parameters—Finite-difference methods.

6. *Propagation problems in continuous systems:* Particular examples—Formulation of the general problem—Mathematical properties—Trial solutions with undetermined parameters—Finite-difference methods for parabolic systems—Finite-difference methods for hyperbolic systems.

One note of criticism concerns the first sentence of the preface: "The advent of high-speed automatic computing machines is making possible the solution of engineering problems of great complexity." However, references in the book to such machines or experience with machines are sparse. This valuable publication certainly did not need this implicit and unfulfilled promise of an introduction to the bright new electronic world.

C. B. T.

1. H. F. P. PURDAY, *Linear Equations in Applied Mechanics*, Interscience Publishers, Inc., New York, 1954. [Rev. 66, *MTAC*, v. 9, p. 131.]

100[X, Z].—GEORGE R. STIBITZ & JULES A. LARRIVEE, *Mathematics and Computers*, McGraw-Hill Book Co., Inc., New York, 1957, vi + 228 p., 23 cm. Price $5.00.

The eleven chapters of this book cover nearly all phases of digital computer design and use. The first three chapters cover the basic elements of computers and mathematics, such as the basic differences between the analog and digital computers and the mathematical definition of a function. These early chapters also include some of the history of mathematics and computing, and outline the types of problems arising in applied mathematics which can be and are solved by computing techniques. The fourth chapter is a discussion of the history of computers which is both interesting and informative but, unfortunately, does not go beyond the ENIAC computer to a description and discussion of the more recent history of stored program computers. The fifth chapter contains discussions of some numerical techniques including those for finding roots of polynomials, solving linear equations, and solving ordinary and partial differential equations. This chapter gives an excellent insight to the beginning student on some of the techniques, but many of the techniques presented are not practical for the modern electronic computer. Bernoulli's method for finding roots of a polynomial equation, for example, is of little importance in the modern computing world. The next three chapters discuss digital computer components, number systems, computer memories, and analog-digital converters. These chapters also outline techniques for performing arithmetic processes in the digital computer, and contain an elementary description of how the computer carries out its operations on the basis of a stored program. The analog computer is likewise treated in these chapters, and the techniques for the mechanical differential analyzer in performing integration are explained. Chapter 9 is a discussion of Monte Carlo techniques and includes an explanation of the solution of linear equations by these techniques. The discussion on sampling techniques includes brief discussions on the solution of certain partial differential equations and techniques for generating "random" sequences. Chapter 10 is a discussion on the various errors which can occur in both analog and digital computers. Chapter 11 summarizes various special-purpose applications of computers such as those seen in the automatic factory, language translation, and, in a lighter vein, computers which play games.

The central difficulty of this reviewer in evaluating this book is in deciding to whom the information is directed. The book seems to be too elementary to serve as a reference book for any computer professional, either user or designer.

On the other hand, the book does not go into sufficient detail in any area to be a successful textbook for most university courses. It could, however, serve as a textbook on an introduction to computers given to freshman or sophomore students or more advanced non-science students. It could also serve as a reference for the beginner in the field or for a non-professional.

The most serious shortcoming of the book is that the material is not sufficiently modern. Many of the numerical techniques presented are old-fashioned and fundamentals are often described in terms of old-fashioned equipment rather than modern equipment. (An analog-digital converter could be easily described in terms of halving and comparing the voltage rather than in terms of the mechanical device which transmits a shaft position to a mechanical digit position.) The authors leave the false impression that tables, especially those involving the elementary functions, are frequently stored in computer memory. Often modern words are not used; for example, the words "subroutine," "programmer," and the term "parity check bit," are not used in describing and discussing these items.

The style of the book is light and enjoyable and makes interesting reading. The bibliography is good and extensive.

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101[X, Z].—W. J. ECKERT & REBECCA JONES, *Faster, Faster*, McGraw-Hill Book Co. Inc., New York, 1955, vii + 160 p., 23 cm. Price $3.75.

Quoting from the preface, "This monograph is an attempt to explain in nontechnical language how a calculator operates, the nature of the problems it solves, and how the problems are presented to the calculator." Actually, it consists almost entirely of a description of "NORC," the Naval Ordnance Research Calculator designed and built by the International Business Machines Corporation. Probably but few experts would agree to the claim that NORC ". . . is also easiest to understand" (preface) yet the explanatory attempt must be judged very successful; it is completely devoid of engineering details (such as component types, circuit diagrams, etc.), relying instead upon simple block diagrams and schematics accompanied by descriptions of the essential properties involved. The language is indeed nontechnical, even such common descriptive contractions as "and/or gate" being avoided, although various colloquialisms native to business machinery, such as "echo-pulse," "print cycle," are explained and used. Here and there the viewpoint tends to be a bit insular; for instance the basic electronic building block turns out to be a binary pulse shaper producing 1 microsecond time delay; this is called a Dynamic Pulse circuit, written always with capitals, like a Thing. Again, no allusion is made to any except IBM equipment, nor to technical contributions by any outsiders except F. C. Williams, the Red Queen, Leibniz and Newton.

Chapter I is introductory, and begins with a general outline of the need for, and requirements of automatic *en masse* arithmetic, and sketches the typical mental, organizational, and instrumental hurdles that must be overcome. Lucid examples of basic arithmetical and scaling operations are given, as well as various notions about electronic components, timing relationships and the rôle of main

functional units such as the arithmetical, memory, input-output, etc. Halfway through the chapter all such generality is jettisoned without rites and supplanted by a clear outline of NORC'S code-language, organization, structure, and philosophy; these and other features of NORC are more fully developed in all chapters but the last of the ten that follow, and in five summarizing appendices. The level of exposition is, on the whole, excellent; one may take exception—as does this reviewer—to the practice of attaching grammatical commas and periods to arithmetical examples, and to the use of the word "accuracy" instead of the word "reliability" in contexts where freedom from accidental errors is meant, but aside from these trivialities of taste, the exposition leaves little to be desired.

NORC is revealed to be quite a fast machine (70 microseconds to multiply; 50 to add or subtract; 250 to divide; 8 for memory access), with ample memory capacity (3600 words, each 66 bits) composed of 4 banks of Williams CRT storage, supplemented by 8 magnetic tape units of impressive capacity (400,000 words each) and speed (4000 words/second), plus printing (19 words/sec) and card punch/read facilities (400 words/min). Each order is an instruction plus three addresses, the latter automatically modifiable en route. The instruction list is almost lavishly flexible; all standard arithmetic operations are available with options of floating point (± 30), automatic sizing shifts, extractions for order modification, transplantation of addressees, etc. There are no less than 21 control transfer instructions, plus 5 for print control and 9 for tape control; the grand total is 98. Data entries are normally preserved to 13 decimal places, and there appears to be an efficient provision for double-precision arithmetic. Beyond doubt, NORC should be classed as an outstandingly flexible and effective computing system.

Electronically, NORC belongs in the category of "pulse recirculating" machines typified by SEAC of the National Bureau of Standards. These machines avoid the use of static memory cells of Eccles-Jordan "flip-flop" type in their arithmetical and logical units by routing binary pulses through various circulating paths consisting of time delay and pulse re-shaping components. Since identification of pulse position on this moving coordinate system is vital, a central clock is used to quantize the time coordinate and great stress is laid upon matters of timing, pulse shaping, and synchronism.

The internal language of NORC consists of words $16\frac{1}{2}$ decimal digits long, each digit being represented by a tetrad of binary digits. It is actually true that arithmetic within the machine is carried out in this mixed base, elements of an essentially binary nature (aggregates of dynamic pulse units plus gangswitches) being combined so as to preserve the local identity (i.e., within tetrads of wires, elements, etc.) of decimal characters. Words are taken from or put into the memory as 66-bit parallel transfers, whereas within the arithmetic unit the basic process involves 4-abreast shifting of successive tetrads to effect, for instance, addition that is decimally serial. Multiplication is done by a similar shift of the multiplicand stepwise through a parallel circuit called a "product generator" equivalent to a multiplication table, so that as each multiplicand digit steps up to the altar, all ten multiples of it become simultaneously available. From this array all multiples specified by the multiplier digits can thus be accumulated at each multiplicand

step, and the entire product assembled during one "pass." Consequently multiplication is almost as brief as addition, though division does not fare comparably well, requiring four or five times as long.

A little bookkeeping may suggest that NORC is somewhat extravagant with regard to information capacity, one binary tetrad capable of representing 16 alternatives being used to represent only 10, etc. This glaring redundancy is the decimal man's burden and seems unavoidable; in NORC some slight byproduct utility is recovered by using "12" and "13" for magnetic tape word-end and block-end signals. Much emphasis is placed upon automatic checking throughout the discussion and NORC uses two systems, (a) modulo 4 summing of the bits in each word, affixing half a tetrad of redundancy to tag this, and (b) "casting-out-nines" checking of the arithmetic operations. Together, (a) and (b) seem to make good sense, whereas (a) alone would be quite weak since repeated doublings play a key rôle in the product generator. In the Williams memory a parity check is also provided for the sum of the bits in each of the 66 parallel positions. Altogether, these features inspire confidence, yet in a machine having some $\frac{1}{3}$ of its capacity redundant, one wonders whether it might not have been feasible to make the checking density far more severe.

The reader concerned with machine design will appreciate that NORC has been moulded to fit rigorously within a framework of precepts: decimal, speedy, ample capacity, big vocabulary, pulsed circuits, checking. Probably not everybody would choose exactly this prescription, but few would deny that it is interesting and bold, and seems to have been carried out with systematic skill and generous material resources, and with the sort of opportunistic ingenuity that is the hallmark of elegant design. The field of computer design is advancing so rapidly both in technology and in concepts that no design group can claim ultimate insight, and all may well benefit from a study of the various species evolved. NORC deserves careful study, and it is to be hoped that a careful comparative examination of its operational performance will eventually be published.

The final chapter in *Faster, Faster* is entitled, "What is there to Calculate?", and consists of a very elementary but clear discussion of calculations applied to linear systems, ballistic trajectores, planetary motion, etc. Perhaps a slight improvement could be made by distinguishing more carefully the idea of solving a mathematical problem in general from the idea of calculating a numerical solution; on page 131 following a discussion of ballistics, the statement that "A slightly more complicated problem is the three-body problem" may illustrate this point. Aside from this, the discussion of numerical procedures is well suited to convey the flavor of this field to the nonspecialist.

J. H. B.

102[F, Z].—D. H. LEHMER, "Sorting cards with respect to a modulus," *J. Assn. for Comp. Machinery*, v. 4, 1957, p. 41–46.

The author gives a method of using an ordinary punched card sorter to separate a deck of punched cards into no more than *m* sub-decks in each of which all numbers are congruent mod *m*.

C. B. T.

103[Z].—L. PEASE, *An Improved Control Board for Card-Programming the IBM 602A Calculating Punch*, Atomic Energy of Canada, Limited, Chalk River, Ontario, A.E.C.L. Report No. 317, 1955, reprinted 1956, vi + 24 + 8 p. of diagrams and figures, 27 cm. Price $1.00.

The paper gives detailed wiring diagrams and instructions for use of a general purpose 602A board. Although the 602A is an electromechanical and, by electronic standards, a slow machine, it has a high degree of flexibility which is utilized in the board described. The board will perform operations on 8-digit numbers of the form: $(A + a)^*(B + b) + c = d$ where * indicates +, −, X or /; the small letters indicate any of seven internal storage registers, and the capital letters fields on the input cards.

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104[Z].—W. G. BROMBACHER, JULIAN F. SMITH, & LYMAN M. VAN DER PYL, *Guide to Instrumentation Literature*, NBS Circular 567, U. S. Gov. Printing Office, Washington, D. C., 1955, iv + 156 p., 26 cm. Price $1.00.

A bibliography listing pertinent works in instrumentation. It includes sections on automatic control, analogue and digital computers, and many other applications more or less related to computation. Books and reference works which appeared no more than about twenty years prior to the compilation of the bibliography were listed, and all periodical articles found were listed. The closing date is not stated, but it was presumably some time during 1954. Sources of material are generally described and listed as specifically as is reasonable.

C. B. T.

105[W, Z].—NATIONAL PHYSICAL LABORATORY, *Wage Accounting by Electronic Computer*, Report No. 1 of the Inter-Departmental Study Group on the Application of Computer Techniques to Clerical Work. Her Majesty's Stationery Office, London, 1956, 25 cm. Price 2s. 6d. net.

This 57-page booklet reports on one of the earliest applications of computers to commercial work in England; the application is a government payroll calculation, and the machine used is the DEUCE. The report is quite complete, including block diagrams, time and cost data, and even considerable discussion of computers per se, reliability, and a section on data sorting. Following are some of the report's conclusions:

(1) A "scientific computer" with magnetic tape input-output forms an adequate machine for payroll work.

(2) The reliability of this operation is satisfactory if attention is given to program checks, proper machine maintenance, and safety margins in the time schedule; automatic "built-in" checking is not essential.

(3) The computer permits reduction in clerical staff, but further economies could be obtained if input data were originally recorded in appropriate form.

(4) The report contains cautious statements indicating that payroll work may justify or at least help justify a computer installation in many organizations.

The DEUCE (successor to ACE) has a 250,000 bit magnetic drum, 32 bit word size, mercury delay line high speed store, and two milliseconds multiply time.

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TABLE ERRATA

The following errata are mentioned in this issue:

CARL-ERIK FRÖBERG, *Hexadecimal Conversion Tables*, Review **82**, p. 208.

H. NAGLER, *Table of Square Roots of Integers*, Review **77**, p. 205.

T. PEARCEY, *Table of the Fresnel Integral to Six Decimal Places*, Review **87**, p. 210–211.

J. RYBNER, *Nomogrammer over komplekse hyperbolski funktioner*, Review **80**, p. 207.

G. N. WATSON, *A Treatise on the Theory of Bessel Functions* [I. M. Longman paper, p. 179].

256.—GEORGE WELLINGTON SPENCELEY, RHEBA MURRAY SPENCELEY, & EUGENE RHODES EPPERSON, *Smithsonian Logarithmic Tables to Base e and Base 10*, The Smithsonian Institution, Washington, D. C., 1952. [Review **992**, *MTAC*, v. 6, 1952, p. 150–151.]

On p. 241 *for* log 1902 = 3,27921 0512*9* 01395 12706
read log 1902 = 3,27921 0512*6* 01395 12706.

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NOTES

Handbook of Mathematical Tables

National Bureau of Standards

The National Science Foundation has commissioned the National Bureau of Standards Applied Mathematics Division to prepare a Handbook of Mathematical Tables containing formulas and graphs. This project is an outgrowth of a conference on Mathematical Tables held at Massachusetts Institute of Technology on September 15 and 16, 1954. One of the principal recommendations made at this conference was that "an outstanding need is for a 'Computer's Handbook,' with usually encountered functions, together with a discussion of their analytic properties and a set of formulas and tables for interpolation and other techniques useful to the occasional computer."

Subsequently Dr. P. A. Smith, Chairman of the Mathematics Department of the National Research Council, appointed a committee composed of A. Erdélyi, M. C. Gray, N. C. Metropolis, P. M. Morse (Chairman), R. D. Richtmeyer, J. B. Rosser, H. C. Thacher, Jr., John Todd, C. B. Tompkins, and J. W. Tukey to advise the NBS staff and help establish a philosophy for the Handbook. After several meetings the following points were established:

1. The proposed preliminary table of contents is that there shall be a general introductory section explaining the use of the tables and the following chapters:

> 1. Powers, Roots and Related Functions; 2. Binomial Coefficients, Bernoulli Numbers; 3. Fundamental Constants; 4. Circular and Hyperbolic Functions, Logarithms; 5. Sine, Cosine, Exponential and Logarithmic Integrals; 6. Gamma and Related Functions; 7. Error Function, Fresnel Integral; 8. Legendre Functions; 9. Bessel Functions including: a) Integral Order, b) Spherical, Modified Spherical, Fractional Order, c) Complex Argument, d) Integrals, e) Struve Functions; 10. Elliptic Functions and Integrals; 11. Mathieu Functions, Spheroidal Wave Functions; 12. Parabolic Cylinder Functions; 13. Hypergeometric Functions; 14. Confluent Hypergeometric Functions; 15. Miscellaneous Functions; 16. Orthogonal Polynomials; 17. Statistical Tables; 18. Interpolation Coefficients—Quadrature; 19. Radix Conversion Tables; 20. Combinatorial Tables.

2. The tables are not to be given to a uniform number of decimal places or significant figures. The elementary functions are to be given to a high order of accuracy because they have basic importance and are essential for the evaluation of higher mathematical functions with the aid of auxiliary functions.

3. The tables will be linearly interpolable to 5D or 5S, as far as possible, although more figures may be given.

4. Polynomial and rational approximations for the various functions are to be given as interpolation and computing aids.

5. Graphs are to be given to demonstrate the behavior of the function or as means of tabulation in the case of some of the higher transcendental functions.

6. Auxiliary functions and arguments will be used extensively to simplify interpolation and permit tabulation over the entire range of the argument.

7. Radix and "Key values" tables to high order of accuracy will be given.

8. In each of the chapters there will be given the most useful mathematical relations pertinent to the functions, indefinite and definite integrals, infinite series, inequalities, and references supplementing the tables.

Part of the computations have already been completed and it is hoped that the volume will be completed in about fifteen months. Dr. M. Abramowitz, of the Applied Mathematics Division of the Bureau of Standards, is in charge of the preparation of the Tables. Queries and suggestions should be addressed to him.

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International Congress of Mathematicians

August 14–21, 1958

The International Congress of Mathematicians will meet in Edinburgh, Scotland from August 14th to August 21st, 1958. The Executive Committee is inviting a number of mathematicians to deliver one-hour and half-hour addresses.

There will also be daily sessions devoted to fifteen-minute communications. There will be eight sections, namely:

1. Logic and Foundations
2. Algebra and Theory of Numbers
3. Analysis
4. Topology
5. Geometry
6. Probability and Statistics
7. Applied Mathematics, Mathematical Physics, and Numerical Analysis
8. History and Education

Those who wish to receive further information about the Congress are requested to communicate their names and full addresses to the Secretary, Frank Smithies at the Mathematical Institute, 16 Chambers Street, Edinburgh 1, Scotland.

(Extracted from preliminary announcement)

N.P.L. Mathematical Table Series

The National Physical Laboratory has announced an N.P.L. Mathematical Table Series to contain tables arising from computational problems received in the Mathematics Division of the N.P.L.

The first volume, *The Use and Construction of Mathematical Tables*, by L. Fox, will be reviewed shortly in *MTAC*.

Forthcoming volumes which have been announced are:

Vol. 2, L. Fox, *Tables of Everett Interpolation Coefficients*, and
Vol. 3, G. F. Miller, *Tables of Generalized Exponential Integrals*.

C. B. T.

CORRIGENDA

Review **44[X, Z]**, *MTAC*, v. 11, 1957, l. 10, p. 48.

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